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INVARIANT MEANS AND SOME MATRIX TRANSFORMATIONS

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Let l_∞, c, c_0 be the Banach spaces of bounded, convergent and null sequences respectively. σ is an injection of the set of positive integers into itself having no finite orbits, and T, the operator defined on l_∞ by $Tx\ (n) = x\ (\sigma n)$. A positive linear functional $\mathcal{L}$ with $\| \mathcal{L} \| = 1$, is called a σ-mean if $\mathcal{L}(x) = \mathcal{L}(Tx)$ for all x in l_∞. A sequence x is said to be σ-convergent, denoted $x \in V_\sigma$, if $\mathcal{L}(x)$ takes the same value, called σ-lim x, for all σ-means. In this paper we characterize the matrices $A \in (l_1, V_\sigma)$ and also study some new sequence spaces l_σ and m_σ.

1. INTRODUCTION

Let σ be a mapping of the set of positive integer into itself. A continuous linear functional $\mathcal{L}$ on l_∞ is said to be an invariant mean or a σ-mean if and only if

(i) $\mathcal{L}\ (x) \geq 0$, when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\mathcal{L}\ (e) = 1$, where $e = \{1, 1, 1, ...\}$

and

(iii) $\mathcal{L}\ (x_{\sigma(n)}) = \mathcal{L}(x)$ for all $x \in l_\infty$.

In case σ is the translation mapping $n \to n + 1$, a σ-mean is often called a Banach limit[2] and V_σ, the set of σ-convergent sequences, that is the set of bounded sequences all of whose invariant means are equal, is the set f of almost convergent sequences[4].

If $x = (x_n)$, set $Tx = (Tx_n) = (x_{\sigma(n)})$. It can be shown[8] that

$$V_\sigma = \{x \in l_\infty : \lim_{m \to \infty} t_{mn}(x) = L, \text{ uniformly in } n, \ L = \sigma\text{-lim } x\}$$

where $t_{mn}(x) = (x_n + Tx_n + \ldots + T^m x_n)/(m + 1)$

and $t_{-1,n} = 0$.

A σ-mean extends the limit functional on c in the sense that $\mathcal{L}(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits, that is to say, if and only if, for all $n \geq 0$, $j \geq 1$, $\sigma^j(n) \neq n$ (see Mursaleen[5]).

A large amount of the literature concerning invariant means etc. can be found in Ahmad and Mursaleen[1], Mursaleen[6] and Raimi[7].

For an infinite matrix $A = (a_{nk})$ and an infinite sequence $x = (x_k)$ of complex numbers, denote $Ax = ((Ax)_n)$, $(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k$, where by the existence of Ax we mean the convergence of this last series for each $n \geq 0$. Given two sequence spaces X and Y, we say that a matrix A is of the type (X, Y) if $Ax \in Y$ whenever $x \in X$.

In section 2 of the present paper, it is shown that V_σ is a Banach space. We also characterize the matrices of the type, (l_1, V_σ), where

$$l_1 = \left\{ x : \sum_{k=0}^{\infty} |x_k| < \infty \right\}.$$

In section 3, we define absolutely σ-convergence and absolutely σ-boundedness and also investigate some properties of these new sequence spaces derived from the concept of invariant means.

2. (l_1, V_σ)-MATRICES

Theorem 2.1 — V_σ is a Banach space normed by

$$||x||_\sigma = \sup_{m,n} |t_{mn}(x)|. \qquad \ldots (1)$$

PROOF : It can easily be verified that (1) actually defines a norm on V_σ and V_σ is a normed linear spaces.

It is left to show that V_σ is complete that is, every Cauchy sequence in V_σ converges to an element of V_σ.

Let $\{x^k\}$ be a Cauchy sequence in V_σ. Then for each i, $\{x_i^k\}_{k=1}^{\infty}$ is a Cauchy sequence in R. Hence $x_i^k \to x_i$ (say) as $k \to \infty$, that is $\lim_k x_i^k = x_i$. Put $x = \{x_i k\}_{i=1}^{\infty}$. By the definition of norm on V_σ it can be easily proved that $\{x^k\} \to x$. It is only left to show that $x \in V_\sigma$.

Since $\{x^k\}$ is a Cauchy sequence, given $\varepsilon > 0$, there exists a positive integer $N(\varepsilon) = N$ (say) such that for each k, $r > N$,

$$||x^k - x^r|| < \varepsilon.$$

Hence, $$\sup_{m,n} |t_{mn}(x^k - x^r)| < \varepsilon.$$

This gives

$$| t_{mn}(x^k - x^r) | < \varepsilon \text{ as } m \to \infty \to | L^k - L^r | < \varepsilon$$

for each m, n and k, $r > N$. Where $L^k = \sigma - \lim x^k$ and let $L = \lim_r L^r$. Then $\mathcal{L}(x) = \lim_k (x^k) = \lim_k L^k = L$. Taking limit as $r \to \infty$, we get

$$| t_{mn}(x^k - x) | < \varepsilon \text{ and } | L^k - L | \leq \varepsilon \qquad \text{... (2)}$$

for each m, n and $k > N$. Now, fix k for which the above inequality holds. Since x^k, for this fixed k, belongs to V_σ, we get

$$\lim_m t_{mn}(x^k) = L^k, \text{ informly in } n.$$

Hence for the given ε, there exists a positive integer $m_0(k)$ such that

$$| t_{mn}(x^k) - L^k | < \varepsilon. \qquad \text{... (3)}$$

for $m \geq m_0(k)$ for all n. Here, $m_0(k)$ is independent of n but depends upon ε. Now, using (2) and (3), we get

$$\begin{aligned} | t_{mn}(x) = L | &= | t_{mn}(x) - t_{mn}(x^k) + t_{mn}(x^k) - L^k + L^k - L | \\ &\leq | T_{mn}(x) - t_{mn}(x^k) | + | t_{mn}(x^k) - L^k | + | L^k - L |. \\ &< 3\,\varepsilon \end{aligned}$$

for $m \geq m_0\ (k)$ and for all n.

This completes the proof of the theorem.

We note that, if, Ax is defined, then for all n, $m \geq 0$.

$$t_{mn}(Ax) = \sum_{k=1}^{\infty} t(n, k, m)\, x_k$$

where

$$t_{n,k,m} = \frac{1}{m+1} \sum_{j=0}^{m} a(\sigma^j(n), k)$$

and $a(n, k)$ denotes the element a_{nk} of the matrix A.

Theorem 2.2 — $A \in (l_1, V_\sigma)$ if and only if

(i) $K = \sup_{n,k,m} | t(n, k, m) | < \infty$

and

(ii) $a_n k_{n=1}^{\infty} \in V_\sigma$ for each k.

PROOF : *Sufficient part* — Suppose $x = (x_k) \in l_1$. By virtue of conditions (i) and

(ii) we see that

$$\lim_{m \to \infty} \sum_{k=1}^{\infty} t(n, k, m)\, x_k = \sum_{k=1}^{\infty} a_k x_k, \text{ uniformly in } n \qquad \text{... (4)}$$

it also converges absolutely. Furthermore, $\sum_k t(n, k, m)\, x_k$ converges absolutely for each m and n.

For a given $\varepsilon > 0$, let $k_0 \in N$, such that

$$\sum_{k > k_0} | x_k | < \varepsilon. \qquad \text{... (5)}$$

By (ii) we can find $m_0 \in N$ such that

$$\left| \sum_{k \le k_0} [t(n, k, m) - a_k]\, x_k' \right| < \varepsilon \qquad \text{... (6)}$$

for all $m > m_0$, uniformly in n. Now,

$$\left| \sum_{k=1}^{\infty} [t(n, k, m) - a_k]\, x_k \right| \le \left| \sum_{k \le k_0} [t(n, k, m) - a_k]\, x_k \right| + \sum_{k > k_0} | t(n, k, m) - a_k | \; | x_k | < (2k + 1)\, \varepsilon$$

for all $m > m_0$ and uniformly in n, by (6), (5) and (i). This proves (4) and hence the sufficiency part.

Necessary Part — Condition (ii) follows from the fact that $e^k \in l_1$. For the necessity of (i), let us define a continuous linear functional $B_{m,n}$ on l_1 by

$$B_{m,n}(x) = \sum_k t(n, k, m)\, x_k.$$

Now, $$| B_{m,n}(x) | \le \sup_k | t(n, k, m) | \; || x ||_1.$$

whence it follows that

$$|| B_{m,n} || \le \sup_k | t(n, k, m) |. \qquad \text{... (7)}$$

For any fixed $k \in N$, define $x = (x_i)$ by

$$x_i = \operatorname{sgn} t(n, k, m), \quad i = k,$$

$$= 0, \; i \ne k.$$

Then $||x||_1 = 1$, and

$$|B_{m,n}(x)| = |t(n,k,m)\, x_k|$$

$$\geq |t(n,k,m)|\ ||x||_1$$

so that

$$||B_{m,n}|| \geq \sup_k |t(n,k,m)|.$$

Using (7), we have

$$||B_{m,n}|| = \sup_k |t(n,k,m)|.$$

Since $A \in (l_1, V_\sigma)$, we have for any $x \in l_1$,

$$\sup_{m,n} |B_{m,n}(x)| = \sup_{m,n} \left| \sum_k t(n,k,m)\, x_k \right| < \infty.$$

Hence, by uniform boundedness principle, we have

$$\sup_{m,n} ||B_{m,n}|| = \sup_{m,n,k} |t_{(n,k,m)}| < \infty.$$

This completes the proof.

If we put $\sigma(n) = n + 1$ in Theorem 2.2, we get the following.

Corollary 2.3 — $A \in (l_1, \hat{f})$, iff

(i) $\sup_{n,k,m} \left| \frac{1}{m+1} \sum_{j=0}^{m} a_{n+j,k} \right| < \infty$

and

(ii) $(a_{nk})_{n=1}^{\infty} \in f$, for each k.

3. Absolutely σ-Convergence

Let $x_n = z_0 + z_1 + \ldots + z_n$, and σ be montonicaly increasing. Write

$$\phi_{mn} = t_{mn}(z) - t_{m-1,n}(z)$$

$$= \frac{1}{m(m+1)} \sum_{j=1}^{m} j \left(\sum_{i=d_j}^{h_j} z_j \right), \quad m \geq 1$$

and $\phi_{0n}(z) = z_n$; with $d_j = \sigma^{j-1}(n) + 1$, $h_j = \sigma^j(n)$. Then we define the spaces l_σ and m_σ, of the absolutely σ-convergent and absolutely σ-bounded sequences respectively, by :

$$l_\sigma = \left\{ z \in V_\sigma : \sum_{m=0}^{\infty} | \phi_{mn}(z) | \text{ converges uniformly in } n \right\}$$

$$m_\sigma = \left\{ z \in l_\sigma : \sup_n \sum_{m=0}^{\infty} | \phi_{mn}(z) | < \infty \right\}.$$

Let $p = (p_m)$ be a sequence of positive real numbers such that $\sup_m p_m < \infty$. We further extend the above sequence spaces to $l_\sigma(p)$ and $m_\sigma(p)$ as follows.

$$l_\sigma(p) := \left\{ z \in V_\sigma : \sum_m | \phi_{mn}(z) |^{p_m} \text{ converges uniformly in } n \right\}$$

and

$$m_\sigma(p) = \left\{ z \in l_\sigma(p) : \sup_n \sum_m | \phi_{mn}(z) |^{p_m} < \infty \right\}$$

Remark : (i) If $p_m = 1$ for all m, we write l_σ and m_σ in place of $l_\sigma(p)$ and $m_\sigma(p)$ respectively.

(ii) If $\sigma(n) = n + 1$, $l_\sigma(p) = \hat{l}(p)$ and $m_\sigma(p) = \hat{l}(p)$, see Das *et al.*[3]

Theorem 3.1 — $l_\sigma(p) \subset m_\sigma(p)$.

PROOF : Suppose that $z \in l_\sigma(p)$. Therefore, for an integer M

$$\sum_{m \geq M} | \phi_{mn}(z) |^{p_m} \leq 1 \qquad \text{... (8)}$$

which gives

$$| \phi_{mn}(z) | \leq 1$$

for $m \geq M$ and all n. But, if $m \geq 1$,

$$\sum_{i = d_m}^{h_m} z_i = (m+1)\phi_{mn} - (m-1)\phi_{m-1,n} \qquad \text{... (9)}$$

and we get each z_i is bounded for any fixed $m \geq M + 1$. Hence ϕ_{mn} is bounded for all m, n. Hence, we get the result.

Theorem 3.2 — $l_\sigma(p)$ and $m_\sigma(p)$ are complete linear spaces paranormed by

$$G_p(z) = \sup_n \left(\sum_m | \phi_{mn}(z) | \right)^{1/H}$$

where $H = \max(1, \sup p_m)$.

The proof is a routine verification by using standard arguments and therefore omitted.

Theorem 3.3 — Suppose that for all m, $p_m \leq q_m$. Then

(i) $l_\sigma(p) \subset l_\sigma(q)$, and

(ii) $m_\sigma(p) \subset m_\sigma(q)$.

PROOF : (i) Let $z \in l_\sigma(p)$. Therefore, there exists an integer M such that, for all n

$$\sum_{m=M}^{\infty} | \phi_{mn}(z) |^{p_m} \leq 1.$$

Hence $| \phi_{mn} | \leq 1$, for $m \geq M$ and all n

so that,

$$| \phi_{mn} |^{q_m} \leq | \phi_{mn} |^{p_m}$$

and the uniform convergence of $\sum | \phi_{mn} |^{q_m}$ follows from that of $\sum_m | \phi_{mn} |^{p_m}$

(ii) Let $z \in m_\sigma(p)$, then

$$\sup_n \sum_m | \phi_{mn} |^{p_m} \leq \infty.$$

Hence, ϕ_{mn} is bounded for all m, n. Therefore there exists $k \geq 1$ such that

$$| \phi_{mn} | \leq K$$

so that

$$\sum_m | \phi_{mn} |^{q_m} \leq \sum_m Kq_m - p_m | \phi_{mn} |^{p_m}$$

$$\leq K^H \sum_m | \phi_{mn} |^{p_m}.$$

where $H = \sup_m q_m$.

Hence $z \in m_\sigma(q)$.

This completes the proof of the theorem.

Remark : Theorem 3.1 and the definition of $m_\sigma(p)$ give that $l_\sigma(p) = m_\sigma(p)$ (see Remark on p. 3 of Savas[9]).

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