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CHARACTERIZATIONS OF BCI/BCH-ALGEBRAS

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Dar *et al.*[1] discussed an (r, l) system, and gave a characterization of BCK and BCI-algebra by such system. In this note, we give a characterization of a BCH-algebra by an (r, l) system. Also we give a characterization of a BCI-algebra by such a system.

Hu and Li[2] introduced the concept of a BCH-algebra. We review some definitions and results.

Definition 1 — A BCI-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the axioms : for every $x, y, z \in X$,

(i) $(x * y) * (x * z) \leq z * y$, ... (1)

(ii) $x * (x * y) \leq y$, ... (2)

(iii) $x \leq x$, ... (3)

(iv) $x \leq y$ and $y \leq x$ imply $x = y$, ... (4)

(v) $x \leq 0$ implies $x = 0$, ... (5)

where $x \leq y$ if and only if $x * y = 0$.

Definition 2[2] — A BCH-algebra is an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following conditions : for every $x, y, z \in X$,

(i) $x \leq x$,

(ii) $x \leq y$ and $y \leq x$ imply $x = y$,

(iii) $(x * y) * z = (x * z) * y$, ... (6)

where $x \leq y$ if and only if $x * y = 0$.

A BCH-algebra has the following basic properties[2].

(i) $x * (x * y) \leq y$,

(ii) $x \leq 0$ implies $x = 0$,

(iii) $x * 0 = x$. ... (7)

It is known that a BCI-algebra is a BCH-algebra, but the converse is not true in general[2].

Definition 3[1] — Let $(X; *, 0)$ be a groupoid with the right identity 0, that is, $x * 0 = x$ for all $x \in X$ and let $\leq$ be an order on X defined by

(i) $x \leq y$ if and only if $x * y = 0$,

(ii) $x \leq y$ and $y \leq x$ imply $x = y$.

Then we say that $(X; *, 0, \leq)$ is an (r, l) system.

Definition 4[1] — For any BCI-algebra/(r, l) system/BCH-algebra X and any element $a \in X$, we use a_r (resp. a_l) to denote the selfmap of X defined by

$$a_r(x) = x * a \text{ (resp. } a_l(x) = a * x)$$

for all $x \in X$.

Proposition 5 — For the selfmaps of a BCH-algebra X, we have the following:

(i) $a_l(0) = 0$ implies $a = 0$ for all $a \in X$, ... (8)

(ii) $a_r a_l = 0_l$ for every $a \in X$, ... (9)

(iii) 0_r is the identity map of X. ... (10)

PROOF : We note that (i) and (iii) above follow immediately from (7).

(ii) Let x be any element of X. Then

$$a_r a_l(x) = a_r(a * x) = (a * x) * a = (a * a) * x = 0 * x = 0_l(x).$$

Hence $a_r a_l = 0_l$ for all $a \in X$.

Now we give a characterization of a BCH-algebra by an (r, l) system.

Theorem 6 — An (r, l) system $(X; *, 0, \leq)$ is a BCH-algebra if and only if

$$(a * b)_l \leq b_r a_l \qquad \text{... (11)}$$

holds for all $a, b \in X$.

PROOF : In view of (6), the necessity is obvious.

Conversely, assume that an (r, l) system $(X; *, 0, \leq)$ satisfies $(a * b)_l \leq b_r a_l$ for all $a, b \in X$. Then for all $x, y, z \in X$, we have

$$(x * y)_l (z) \leq (y_r x_l) (z),$$

that is,

$$(x * y) * z \leq (x * z) * y. \qquad \text{... (12)}$$

Since x, y and z are arbitrary, therefore, by (4), it follows that

$$(x * y) * z = (x * z) * y.$$

If we put $y = 0 = z$ in (2), we obtain

$$(x * 0) * 0 \leq (x * 0) * 0.$$

Hence by (7), we have $x \leq x$. This completes the proof of the theorem.

A characterization of a BCI-algebra by an (r, l) system was given as follows[1] :

Proposition 7[1] — An (r, l) system $(X;\ *,\ 0,\ \leq)$ is a BCI-algebra if and only if

$$(a * b)_r\,(b_r * a_r) = 0$$

for all $a, b \in X$.

Now we give another characterization of a BCI-algebra by an (r, l) system.

Theorem 8 — Let X be a BCH-algebra satisfying

$$a \leq b \text{ implies } a_l \leq b_l \qquad \text{... (13)}$$

for all $a, b \in X$. Then X is a BCI-algebra.

PROOF : It suffices to prove that (1) is true. By (2), we have $x * (x * z) \leq z$. Therefore by (13), we obtain $(x * (x * z)) * y \leq z * y$. Hence by (6), we have $(x * y) * (x * z) \leq z * y$. This completes the proof.

Combining Theorems 6 and 8, we obtain a characterization of a BCI-algebra.

Theorem 9 — An (r, l) system $(X;\ *,\ 0,\ \leq)$ is a BCI-algebra if and only if it satisfies the conditions (11) and (13).

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