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FORCED OSCILLATIONS OF HYPERBOLIC DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Sufficient conditions are established for the oscillation of some hyperbolic equations with deviating arguments under boundary conditions of Dirichlet, Nuemann and Robin type.

1. INTRODUCTION

Recently there has been a growing interest towards the study of qualitative properties of partial differential equations with deviating arguments. However, only a few results have appeared so far which deal with the oscillation of hyperbolic equations with deviating arguments. We refer to the contributions by Georgiou and Kreith[1], Mishev[2], Mishev and Bainov[3,4], Yoshida[5] and the references cited therein.

The present paper is concerned with forced oscillation of the solutions of hyperbolic equations of the form

$$\text{(E)}\quad \frac{\partial^2}{\partial t^2} u(x, t) = a(t)\, \Delta u\,(x, t) + \sum_{i=1}^{m} a_i\,(t)\, \Delta u\,(x, \rho_i\,(t))$$

$$- \sum_{j=1}^{k} p_j\,(x, t)\, u(x, \sigma_j(t)) + f(x, t),\ \ (x, t) \in \Omega \times (0, \infty) \equiv G$$

where Δ is the Laplacian in the Euclidean n-space R^n, and Ω is a bounded domain in R^n with a smooth boundary $\delta\Omega$.

Differential equations with deviating arguments e.g., with delay in terms in the timelike variables, provide a mathematical model for physical system in which the rate of change of the system depends on its past history. It is well known that a differential equation with delay and the corresponding differential equation without delay do not always exhibit the same oscillatory properties. The equation $u''(t) = -u(t-\pi) = 0$ has the oscillatory solutions $\sin t$ and $\cos t$ although the corresponding ordinary equation $u''(t) - u(t) = 0$ is nonoscillatory. It is known that, in some cases, the introduction of delays does not change the qualitative behaviour of the physical model. The equation $y''(t) + q(t)y^{\gamma}(t) = 0$ is oscillatory provided $\gamma > 1$ and $\int^{\infty} sq(s)\, ds = \infty$, where γ is the ratio of positive odd integers. The delay equation $y''(t) + q(t)y^{\gamma}(t - \tau(t)) = 0$ is also oscillatory under the same conditions, where $\tau(t)$ is assumed to be a positive, continuous function on $[0, \infty)$ which, for some M, satisfies $0 < \tau(t) \leq M$. Thus oscillations caused by delays should be seriously taken into account when we study physical mechanism, especially, when it is moving with high speed where a sudden release of oscillations may lead to instability of the mechanism.

We list a few assumptions :

(A_1) $a, a_j \in C(R_+, R_+)$, $R_+ = (0, \infty)$ $j = \{1, 2, ..., k\}$ and $f \in C(\overline{\Omega} \times R_+, R)$;

(A_2) σ_i, $\rho_j \in C(R_+, R)$, $\lim_{t\to\infty} \sigma_j(t) = \lim_{t\to\infty} \rho_i(t) = \infty$;

$i = 1, 2, ..., m$, $j = 1, 2, ..., k$;

(A_3) $p_j \in C(\overline{\Omega} \times R_+, R)$, $p_j(x, t) \geq 0$, and $p_j(t) = \min_{x\in\overline{\Omega}} \{ p_j(x, t) \}$, $j = 1, 2, ..., k$.

Our aim is to establish sufficient conditions under which every (classical) solution u of (E) satisfying a certain boundary condition is oscillatory on $\Omega \times [0, \infty)$ in the sense that u has a zero on $\Omega \times [t, \infty)$ for every $t > 0$. We consider three kinds of boundary conditions :

(B_1) $\dfrac{\partial u}{\partial \nu} + \mu u = 0$ on $\partial\Omega \times R_+$,

(B_2) $u = \phi$ on $\partial\Omega \times R_+$;

(B_3) $\dfrac{\partial u}{\partial \nu} = \psi$ on $\partial\Omega \times R_+$;

where ϕ, ψ, μ are continuous and real valued on $\partial\Omega \times R_+$, ν denotes the unit exterior normal vector to $\partial\Omega$ and $\mu(x, t) \geq 0$ on $\partial\Omega \times R_+$.

2. Oscillation of Problem (E), (B_1)

We first prove a lemma which is useful for results in this section :

Lemma 2.1 — Suppose that $y \in C^2([t_0, \infty), R))$ and that

$$y(t) > 0,\ y'(t) > 0 \text{ and } y''(t) \leq 0,\ t \geq t_0 > 0. \quad \text{... (2.1)}$$

Then for any $\lambda_0 \in (0, 1)$ there exists a number $t_1 > t_0$ such that

$$y(t) \geq \lambda_0 t y'(y) \quad \text{for } t \geq t_1. \qquad \text{... (2.2)}$$

PROOF : There exists a number ξ such that $y(t) - y(t_0) = y'(\xi)(t - t_0)$, $\xi \in (t_0, t)$. It follows from (2.1) that

$$y(t) \geq y'(t)(t - t_0) \quad \text{for } t \geq t_0. \qquad \text{... (2.3)}$$

For $\lambda_0 \in (0, 1)$ we put $\mu = \dfrac{1}{1-\lambda_0} > 1$. Then $\lambda_0 = 1 - \dfrac{1}{\mu}$ and hence

$$t - t_0 \geq t - \frac{t}{\mu} = t\left(1 - \frac{1}{\mu}\right) = \lambda_0 t \quad \text{for } t \geq \mu t_0 = t_1. \qquad \text{... (2.4)}$$

Thus (2.4) and (2.3) yield (2.2).

Theorem 2.1 — Suppose that conditions $(A_1) - (A_3)$ hold and that :

(A_4) There exists an oscillatory function $F \in C^2(R, R)$ such that

$$F''(t) = \int_\Omega f(x, t)\, dx \text{ and } \lim_{t \to \infty} F(t) = 0.$$

Then every solution u of the problem (E), (B_1) is oscillatory in G if the delay differential inequality

$$y''(t) + \lambda_0 \sum_{j=1}^{k} p_j(t) y(\sigma_j(t)) \leq 0 \qquad \text{... (2.5)}$$

has no eventually positive solutions for every $\lambda_0 \in (0, 1)$.

PROOF : Let u be a nonoscillatory solution of (E) and (B_1) which we may assumed (and do) to be positive on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. In view of A_2 there is a number $t_1 \geq t_0$ such that $u(x, \rho_i(t)) > 0$, $u(x, \sigma_j(t)) > 0$; $i = 1, 2, ..., n$, $j = 1, 2, ..., k$ and for all $(x, t) \in \Omega \times [t_1, \infty)$. From Green's formula and conditions (A_3) and (B_1) we have

$$\frac{d^2}{dt^2}\left[\int_\Omega u(x, t)\, dx\right] = a(t)\int_\Omega \Delta u(x, t)\, dx + \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u(x, \rho_i(t))\, dx$$

$$- \sum_{j=1}^{k} \int_\Omega p_j(x, t)\, u(x, \sigma_j(t))\, dx + \int_\Omega f(x, t)\, dx$$

$$\leq a(t) \int_{\delta\Omega} [-\mu u]\, d\omega + \sum_{i=1}^{m} a_i(t) \int_{\delta\Omega} [-\mu(x, \rho_i(t))\, u(x, \rho_i(t))]\, d\omega$$

$$- \sum_{j=1}^{k} p_j(t) \int_\Omega u(x, \sigma_j(t))\, dx + \int_\Omega f(x, t)\, dx$$

$$\leq -\sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx + \int_{\Omega} f(x, t)\, dx, \quad t \geq t_1. \qquad \text{... (2.6)}$$

Set

$$y(t) = \int_{\Omega} u(x, t)\, dx - F(t), \quad t \geq t_1. \qquad \text{... (2.7)}$$

It follows from (2.6) that

$$y''(t) + \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx \leq 0, \quad t \geq t_1. \qquad \text{... (2.8)}$$

We claim that there is a number $t_2 \geq t_1$ such that the function y satisfies the condition (2.1) for $t \geq t_2$. In fact, if $y(t) \leq 0$ then $\int_{\Omega} u(x, t)\, dx \leq F(t)$. Which is impossible in view the fact that $u(x, t) > 0$ and the function F is oscillatory. From (2.8) we have $y''(t) \leq 0$, $t \geq t_2$. Using the fact that $y(t) > 0$ and $y''(t) \leq 0$ we have $y'(t) > 0$, $t \geq t_2$. Now, since y is an increasing function and $\lim_{t \to \infty} F(t) = 0$ it follows from (2.7) that there are numbers $t_3 \geq t_2$ and $\lambda_0 \in (0, 1)$ such that

$$\int_{\Omega} u(x, t)\, dx \geq \lambda_0 y(t) \text{ and } \int_{\Omega} u(x, \sigma_j(t))\, dx \geq \lambda_0 y(\sigma_j(t)), \; t \geq t_3, \; j = 1, 2, ..., k.$$

Consequently, we get

$$y''(t) + \lambda_0 \sum_{j=1}^{k} p_j(t) y(\sigma_j(t)) \leq 0, \quad t \geq t_3$$

which contradicts the assumption that (2.5) has no eventually positive solution. In case $u(x, t) < 0$ then the function $\bar{u}(x, t) = -u(x, t)$ is a positive solution of the problem

$$\frac{\partial^2}{\partial t^2} u(x, t) = a(t)\, \Delta u(x, t) + \sum_{i=1}^{m} a_i(t)\, \Delta u(x, \rho_i(t))$$

$$- \sum_{j=1}^{k} p_j(x, t)\, u(x, \sigma_j(t)) - f(x, t), \; (x, t) \in \Omega \times (0, \infty) \equiv G$$

$$\frac{\partial}{\partial \nu} u + \mu u = 0, \text{ on } \partial\Omega \times (0, \infty).$$

Now set $y(t) = \int_{\Omega} \bar{u}(x, t)\, dx - F(t)$, $t \geq t_0$ and use an argument similar to the one used earlier to arrive at a contradiction. This completes the proof.

Theorem 2.2 — Suppose that conditions $(A_1) - (A_4)$ hold and that

(A_5) $\sigma(t) = \max_{1 \le j \le k} \{ \sigma_j(t) \} \le t$, $\sigma'(t) \ge 0$.

If for every number $\lambda_0 \in (0, 1)$

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(t)\, ds > \frac{1}{\lambda_0^2} \qquad \text{... (2.9)}$$

then every solution u of problem (E), (B_1) is oscillatory on G.

PROOF : On the contrary let u be a nonoscillatory solution of (E), (B_1) which we assume to be positive. The function y defined by (2.7) satisfies the inequalities (2.1) and (2.5) for some $\lambda_0 \in (0, 1)$. By Lemma 2.1 we can choose a number t_1 sufficiently large such that

$$y(t) \ge \lambda_0\, t y'(t) \ \text{ for } t \ge t_1$$

and

$$y(\sigma_j(t)) \ge \lambda_0\, \sigma_j(t)\, y'(t) \quad \text{for } t \ge t_1,\ j = 1, 2, \ldots, k.$$

Now we use (2.5) to get

$$y''(t) + \lambda_0^2 \sum_{j=1}^{k} p_j(t)\, \sigma_j(t)\, y'(\sigma_j(t)) \le 0, \ t \ge t_1.$$

Integrating the above inequality from $\sigma(t)$ to t we have

$$y'(t) - y'(\sigma(t)) + \lambda_0^2 \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, y'(\sigma_j(t)) \le 0, \quad t \ge t_1$$

Therefore,

$$\lambda_0^2 \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds \le 1 - \frac{y'(t)}{y'(\sigma(t))} < 1.$$

And hence

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds \le \frac{1}{\lambda_0^2}$$

which violates the condition (2.9). The proof the case $u < 0$ is similar and is omitted.

Corollary 2.1 — In addition to conditions (A_1)-(A_3) let (A_5) hold and suppose that $f(x, t) \equiv 0$. If

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds > 1$$

then every solution u of the problem (E), (B_1) is oscillatory on G.

3. Oscillation of Problem (E), (B_2)

It is known that the least eigenvalue α_1 of the problem

$$\Delta u + \alpha u = 0\,, \quad u \in \Omega, \qquad u = 0 \text{ on } \partial\Omega. \qquad \text{... (3.1)}$$

is positive and the corresponding eigenfunction Φ is positive on Ω.

We prove the following

Theorem 3.1 — Assume conditions (A_1)-(A_3) and let

(A_6) there exists an ~~oscillatory~~ function $H \in C^2(R, R)$ with $\lim_{t\to\infty} H(t) = 0$ and which satisfies

$$H''(t) = \int_\Omega f(x, t)\,\Phi(x)\,dx - \int_{\partial\Omega} \left[a(t)\,\phi(x, t) + \sum_{i=1}^{m} a_i(t)\,\phi(x, \rho_i(t)) \right] \frac{\partial\Phi}{\partial\nu}\,d\omega$$

for all large t. If the delay differential inequality

$$y''(t) + \lambda \left\{ \alpha_1\,a(t)\,y(t) + \alpha_1 \sum_{i=1}^{m} a_i(t)\,y(\rho_i(t)) + \sum_{j=1}^{k} p_j(t)\,y(\sigma_j(t)) \right\} \le 0 \qquad \text{... (3.2)}$$

for every $\lambda \in (0, 1)$ then every solution of the problem (E), (B_2) is oscillatory on G.

Proof : Suppose that there is nonoscillatory solution u of (E), (B_2) which, as before, we assume to be positive on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. In view of (A_2) there is a number $t_1 \ge t_0$ such that $u(x, t) > 0$, $u(x, \sigma_j(t)) > 0$, $i = 1, 2, ..., m$, $j = 1, 2, ..., k$ for all $(x, t) \in \Omega \times [t_1, \infty)$. It follows from (E) that

$$\frac{\partial^2}{\partial t^2} u(x, t) \le a(t)\,\Delta u(x, t) + \sum_{i=1}^{m} a_i(t)\,\Delta u\,(x, \rho_i(t))$$
$$- \sum_{j=1}^{k} p_j(t)\,u(x, \sigma_j(t)) + f(x, t)\,, \quad (x, t) \in \Omega \times [t_1, \infty). \qquad \text{... (3.3)}$$

Multiplying (3.3) by Φ and integrating over Ω we obtain

$$\frac{d^2}{dt^2}\left[\int_\Omega u(x, t)\,\Phi(x)\,dx \right] \le a(t) \int_\Omega \Delta\,u\Phi(x)\,dx$$
$$+ \sum_{i=1}^{m} a_i(t) \int_\Omega \Delta u(x, \rho_i(t))\,\Phi(x)\,dx$$

$$-\sum_{k=1}^{k} p_i(t) \int_\Omega u(x, \sigma_j(t))\, \Phi(x)\, dx + \int_\Omega f(x, t)\, \Phi(x)\, dx \qquad \text{... (3.4)}$$

for $t \geq t_1$. Using Green's Theorem we have

$$\int_\Omega \Delta u \Phi(x)\, dx = \int_{\partial\Omega} \left(\Phi \frac{\partial u}{\partial \nu} - u \frac{\partial \Phi}{\partial \nu} \right) d\omega + \int_\Omega u \Delta \Phi\, dx$$

$$= -\int_{\partial\Omega} \phi \frac{\partial \Phi}{\partial \nu}\, d\omega - \alpha_1 \int_\Omega u(x, t)\, \Phi(x)\, dx, \; t \geq t_1 \qquad \text{... (3.5)}$$

$$\int_\Omega \Delta u(x, \rho_i(t))\, \Phi(x)\, dx = \int_{\partial\Omega} \left(\frac{\partial u(x, \rho_i(t))}{\partial \nu} \Phi(x) - u(x, \rho_i(t)) \frac{\partial \Phi}{\partial \nu} \right) d\omega$$

$$+ \int_\Omega u(x, \rho_i(t))\, \Delta \Phi\, dx$$

$$= -\int_{\partial\Omega} \phi(x, \rho_i(t)) \frac{\partial \Phi}{\partial \nu}\, d\omega - \alpha_1 \int_\Omega u(x, \rho_i(t))\, \Phi(x)\, dx, \; t \geq t_1. \qquad \text{... (3.6)}$$

Let $$V(t) = \int_\Omega u(x, t)\, \Phi(x)\, dx, \; t \geq t_0. \qquad \text{... (3.7)}$$

Now combine (3.4)-(3.7) to get

$$V''(t) \leq -\alpha_1 \left\{ a(t) V(t) - \sum_{i=1}^{m} a_i(t) V(\rho_i(t)) \right\} - \sum_{j=1}^{k} p_j(t) V(\sigma_j(t))$$

$$- a(t) \int_{\partial\Omega} \phi(x, t) \frac{\partial \Phi}{\partial \nu}\, d\omega - \sum_{i=1}^{m} a_i(t) \int_{\partial\Omega} \phi(x, \rho_i(t)) \frac{\partial \Phi}{\partial \nu}\, d\omega$$

$$+ \int_\Omega f(x, t)\, \Phi(x)\, dx, \; t \geq t_1. \qquad \text{... (3.8)}$$

Now with

$$y(t) = V(t) - H(t), \; t \geq t_1$$

the inequality (3.8) becomes

$$y''(t) + \alpha_1 \left\{ a(t) V(t) + \sum_{i=1}^{m} a_i(t) V(\rho_i(t)) \right\} + \sum_{j=1}^{k} p_j(t) V(\sigma_j(t)) \leq 0, \; t \geq t_1. \qquad \text{... (3.9)}$$

We note that $V > 0$ hence as in the proof of Theorem 2.1 we have

$$y'(t) > 0, \qquad t \geq t_2 \geq t_1, \qquad \text{... (3.10)}$$

and from (2.8) we have

$$y''(t) \le 0, \quad t \ge t_1. \qquad \text{... (3.11)}$$

From (3.10) and (3.11) it follows that $y'(t) > 0$, $t \ge t_1$. Since y is an increasing function and $\lim_{t \to \infty} H(t) = 0$ we conclude from $V(t) = y(t) + H(t)$ that there exists a number $t_3 \ge t_2$ and a number $\lambda_1 \in (0, 1)$ such that the following inequalities hold :

$$V(t) \ge \lambda_1 y(t), \quad t \ge t_3$$

$$V(\sigma_j(t)) \ge \lambda_1 y(\sigma_j(t)), \ t \ge t_3, \ j = 1, 2, ..., k,$$

$$V(\rho_i(t)) \ge \lambda_1 y(\rho_i(t)), \ t \ge t_3, \ i = 1, 2, ..., m.$$

Thus in view of (3.9) the inequality (3.2) has an eventually positive solution which is a contradiction. A similar proof can be given for the case $u < 0$. This completes the proof of Theorem 3.1.

The proofs of the following two Theorems can be modelled on that of Theorem 3.1.

Theorem 3.2 — Suppose that the conditions (A_1)-(A_3) and (A_6) hold. If

$$\limsup_{t \to \infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds > \frac{1}{\lambda^2}, \text{ for every } \lambda \in (0, 1),$$

then every solution of the problem (E), (B_2) is oscillatory in G.

Theorem 3.3 — In addition to conditions (A_1)-(A_3) and (A_6) suppose that

(A_7) $\qquad \rho(t) = \max_{1 \le i \le m} \{\rho_j(t)\} \le t, \quad \rho'(t) \ge 0.$

If

$$\lim \sup_{t \to \infty} \int_{\rho(t)}^{t} \sum_{i=1}^{m} a_i(s)\, \rho_i(s)\, ds > \frac{1}{\lambda^2 \alpha_1} \text{ for every } \lambda \in (0, 1)$$

then every solution u of the problem (E), (B_2) is oscillatory.

4. Oscillation for Problem (E), (B_3)

Now we establish some oscillation criteria for problem (E), (B_3) and give some illustrative examples.

Theorem 4.1 — In addition to conditions (A_1)-(A_3) assume that

(A_8) There exists an oscillatory function $H(t) \in C^2(R, R)$ satisfying $\lim_{t \to \infty} H(t) = 0$ and such that

$$H'(t) = \int_{\Omega} f(x, t)\, dx + \int_{\partial\Omega} \left(a(t)\, \phi(x, t) + \sum_{i=1}^{m} a_i(t)\, \psi(x, \rho_i(t)) \right) d\omega.$$

If the delay differential inequality

$$y''(t) + \lambda \sum_{j=1}^{k} p_j(t)\, y(\sigma_j(t)) \le 0 \text{ for every } \lambda \in (0, 1) \quad \text{... (4.1)}$$

has no eventually positive solutions then every solution of problem (E), (B_3) is oscillatory on G.

PROOF : Suppose that there is a nonoscillatory solution u of the problem. As before we assume that $u > 0$ on $\Omega \times [t_0, \infty)$ for some $t_0 > 0$. We can find a number $t_1 \ge t_0$ such that $u(x, \rho_i(t)) > 0$, $u(x, \sigma_j(t)) > 0$, $i = 1, 2, ..., m$; $j = 1, 2, ..., k$ for all $(x, t) \in \Omega \times [t_1, \infty)$. Now we use Green's formula and integrate (E) over G to get

$$\frac{d^2}{dt^2}\left(\int_{\Omega} u(x, t)\, dx \right) \le a(t) \int_{\Omega} \Delta u(x, t)\, dx + \sum_{i=1}^{m} a_i(t) \int_{\Omega} \Delta\, u(x, \rho_i(t))\, dx$$

$$- \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx + \int_{\Omega} f(x, t)\, dx$$

$$= a(t) \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\omega + \sum_{i=1}^{m} a_i(t) \int_{\partial\Omega} \frac{\partial u\, (x, \rho_i(t))}{\partial \nu}\, d\omega$$

$$- \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx + \int_{\Omega} f(x, t)\, dx$$

$$= a(t) \int_{\partial\Omega} \psi(x, t)\, d\omega + \sum_{i=1}^{m} a_i(t) \int_{\partial\Omega} \psi(x, \rho_i(t))\, d\omega$$

$$- \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx + \int_{\Omega} f(x, t)\, dx, \quad t \ge t_1. \quad \text{... (4.2)}$$

It follows from (A_5) and (4.2) that

$$\frac{d^2}{dt^2}\left(\int_{\Omega} u(x, t)\, dx - H(t) \right) + \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(t))\, dx \le 0, \quad t \ge t_1. \quad \text{... (4.3)}$$

We let

$$y(t) = \int_{\Omega} u(x, t)\, dx - H(t) \quad \text{... (4.4)}$$

and use it along with (4.3) to realize

$$y''(t) + \sum_{j=1}^{k} p_j(t) \int_{\Omega} u(x, \sigma_j(T))\, dx \geq 0, \qquad t \geq t_1.$$

The remainder of the proof is similar to that of Theorem 2.1 and is omitted.

The proof of the following theorem is similar to that of Theorem 2.2.

Theorem 4.2 — Suppose that conditions (A_1) – (A_3), (A_5) and (A_6) hold. If

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds > \frac{1}{\lambda^2} \text{ for every } \lambda \in (0, 1)$$

then the conclusion of Theorem 4.1 holds.

Corollary 4.1 — Suppose that conditions (A_1)-(A_3) and (A_5) hold, $f(x, t) = 0$ and that there exists an oscillatory function $\eta(t) \in C^2(R, R)$ with $\lim_{t\to\infty} \eta(t) = 0$ and

$$\eta''(t) = \int_{\partial\Omega} \left(a(t)\, \psi(x, t) + \sum_{i=1}^{m} a_i(t)\, \psi(x, \rho_i(t)) \right) d\omega.$$

If

$$\limsup_{t\to\infty} \int_{\sigma(t)}^{t} \sum_{j=1}^{k} p_j(s)\, \sigma_j(s)\, ds > \frac{1}{\lambda^2} \text{ for every } \lambda \in (0, 1).$$

Then the conclusion of Theorem 4.1 holds.

Example 1 — Consider the hyperbolic equation

$$(E_1)\ u_{tt} = u_{xx}(x, t-\pi) - e^{-t}\, u(x, t-\pi) - u(x, t-2\pi)$$
$$- e^{-t}(1 + \cos x) \cos t, \qquad (x, t) \in (0, \pi) \times (0, \infty),$$

with boundary condition

$$(B_4) \ - \ u_x(0, t) = u_x(\pi, t) = 0, \qquad t \geq 0.$$

Here $a(t) = 1$, $p_1(t) = e^{-t}$, $p_2(t) = 1$, $f(x, t) = -e^{-t}(1 + \cos x) \cos t$, $\sigma(t) = \max \{t-\pi, t-2\pi\} = t - \pi \leq t$. We note that

$$\int_{\Omega} f(x, t)\, dx = \int_{0}^{\pi} e^{-t}(1 + \cos x) \cos t\, dx = -\pi\, e^{-t} \cos t. \qquad \text{... (4.5)}$$

We can choose $F''(t) = \frac{\pi}{2} e^{-t} \cos t$. It is easy to verify that all the hypotheses of Theorem 2.2 are satisfied and hence all the solutions of problem (E_1), (B_4) are oscillatory. One such solution is $u(x, t) = (1 + \cos x) \cos t$.

Example 2 — Consider the hyperbolic equation

$$(\mathrm{E}_2)\quad u_{tt} = u_{xx} + e^{\pi} u_{xx}(x, t-\pi) + e^{\pi/2} u_{xx}\left(x, t-\frac{\pi}{2}\right)$$

$$- e^{\pi/2} u\left(x, t-\frac{\pi}{2}\right) e^{\pi-2t} u(x, t-\pi) - e^{-t}\cos t \sin x,$$

for all $(x, t) \in (0, \pi) \times (0, \infty)$, with boundary condition

$$(\mathrm{B}_5)\quad u(0, t) = u(\pi, t) = 0,\ t \geq 0.$$

Here $\Omega = (0, \pi)$, $a(t) = 1$, $a_1(t) = e^{\pi}$, $a_2(t) = e^{\pi/2} = p_1(t)$, $p_2(t) = e^{\pi-2t}$, $\sigma_1(t) = t - \frac{\pi}{2}$, $\sigma_2(t) = t - \pi$ and $\sigma(t) = t - \frac{\pi}{2}$. Moreover, the corresponding eigenvalue problem

$$\Delta u + \alpha u = 0,\ x \in (0, \pi) \text{ and } u = 0,\ x = 0, \pi$$

has the eigenvalue $\alpha_1 = 1$ with the corresponding eigenfunction $\Phi(x) = \sin x > 0$ on $(0, \pi)$. We note that

$$\int_{\Omega} f(x, t)\,\Phi(x)\,dx = -\int_0^{\pi} e^{-t}\cos t \sin^2 x\,dx = -\frac{\pi}{2} e^{-t}\cos t. \quad \text{... (4.6)}$$

Choose the function $H(t) = \frac{\pi}{4} e^{-t}\sin t$. Now it is easily checked that the hypotheses of Theorem 3.2 are verified. Thus all the solutions of problem (E_2), (B_5) are oscillatory. One such solution is $u(x, t) = e^{t}\cos t \sin x$.

Remark : The work in this paper may be considered a generalization of work of Georgou and Kreith[1] who consider only hyperbolic equations in canonical form with constant delay terms.

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