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A PRODUCT FORMULA FOR COBCAT AND SOME CALCULATIONS

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The objects of this paper are (i) to obtain a product formula for cobordism category of singular mainfolds and G-manifolds, and (ii) to calculate the cobordism category of a few classes of Dold manifolds.

1. INTRODUCTION

Singh[5] developed a notion called the cobordism category of manifolds. We have extended this notion to singular manifolds and G-manifolds, G being a compact Lie group[1,2]. Here, in section 3, we have obtained a product formula for cobordism category of singular manifolds and G-manifolds, and in section 4, we have calculated the cobordism category of a few classes of Dold manifolds.

Here all the manifolds are to be unoriented, smooth and closed, and all the singular homology and cohomology coefficients are to be in $\mathbf{Z}_2$.

2. PRELIMINARIES

Let (M^n, f) be a singular manifold in a space X. Then the cobordism category of (M^n, f), denoted by cobcat (M^n, f), is the smallest positive integer k such that for each m, $0 \leq m \leq n$, the number

$$\langle W_{i_1} \ldots W_{i_p} f^* (x_{j_1} \ldots x_{j_q}),\ [M^n] \rangle = 0$$

for all partitions $i_1 + \ldots + i_p = m$ and $j_1 + \ldots + j_q = n - m$ with $k \leq p + q \leq n$ $(x_{j_h} \in H^{j_h}(x))$ for all j_h); if no such k exists then set cobcat $(M^n, f) = n + 1$. Note that if f is a constant map then the notion of cobcat (M^n, f) coincides with that of cobcat (M^n) introduced by Singh[5]. Again if G is a compact Lie group and $(M^n; G)$ is a G-manifold with G acting freely then the cobordism category of $(M^n; G)$, denoted by G-cobcat (M^n), is the smallest positive integer k such that the number $\langle \tau_M^* (x_{i_1} \ldots x_{i_p}),\ [M^n; G] \rangle = 0$ for all partitions $i_1 + \ldots + i_p = n\text{-dim } G$ with $k \leq p \leq n-$

dim G and for all $x_{i_j} \in h^{i_j}(B(O,G);G)$; if no such k exists then set G-cobcat $(M^n) = n - \dim G + 1$ (here $\tau_M : M^n \to B(O,G)$ denotes the classifying map of the G-tangent bundle over M^n).

Remark 2.1 : G-cobcat (M^n) = cobcat $(M^n/G, f)$, where $f : M^n/G \to BG$ classifies the principal G-bundle $M^n \to M^n/G$, (cf. Das and Khare[2]).

Further, suppose $k_i : X_i \to BG$ be any map on a topological space X_i, $i = 1, 2$. Consider the quotient $Ek_1^*(\gamma) . Ek_2^*(\gamma) = (Ek_1^*(\gamma) \times Ek_2^*(\gamma))/\sim$, where $(x, y) \sim (g^{-1}x, gy)$, $g \in G$, $x \in Ek_1^*(\gamma)$, $y \in Ek_2^*(\gamma)$; $\gamma : EG \to BC$ being the universal G-bundle. Define an a action is free. Moreover, one can see that the map $(Ek_1^*(\gamma) . Ek_2^*(\gamma))/G \to X_1 \times X_2$, given by $[[x, y]] \to (p_1(x), p_2(y))$, is a homeomorphism; p_i being the projection of the bundle $k_i^*(\gamma)$, $i = 1, 2$. Denote the principal G-bundle $Ek_1^*(\gamma) . Ek_2^*(\gamma) \to X_1 \times X_2$ by $k_1^*(\gamma) . k_2^*(\gamma)$. In particular, considering $X_1 = X_2 = BG$ and $k_1 = k_2 = l_{BG}$, we get a principal G-bundle $\gamma . \gamma$ over $BG \times BG$. Let $\mu_{BG} : BG \times BG \to BC$ be the classifying map of $\gamma . \gamma$. One can see that μ_{BG} is an H-space structure in BG, (Das and Khare[2]). Then if $(M^m; G)$ and $(N^n; G)$ are two G-manifolds, their product may be defined as

$$(M^m ; G) \times (N^n ; G) = (E((\mu_{BG} \circ (f_M \times f_N))^*(\gamma)) ; G)$$

$$= (E((f_M \times f_N)^*(\gamma . \gamma)) ; G) = (Ef_M^*(\gamma) . Ef_N^*(\gamma) ; G)$$

$$= (M^m . N^n ; G)$$

where $f_M : M^m/G \to BG$ and $f_N : N^n/G \to BG$ are the classifying maps of the principal C-bundles $M^m \to M^m/G$ and $N^n \to N^n/G$ respectively. Note that this product induces an algebra structure in N_*^G such that the N_*-module isomorphism $N_*^G \cong N_*(BG)$ given by $[M^m ; G] \to [M^m/G, f_M]$ becomes an algebra isomorphism. Here the product structure in $N_*(BG)$ is defined as $[M^m, f] \times [N^n, g] = [M^m \times N^n, \mu_{BG} \circ (f \times g)]$ where (M^m, f) and (N^n, g) are singular manifolds in BG.

3. Product Formula for Cobcat

Let X and Y be two topological spaces, and (M^m, f) and (N^n, g) be two singular manifolds in X and Y respectively. Then the product $(M^m \times N^n, f \times g)$ is a singular manifold in $X \times Y$.

Proposition 3.1 — With notations as above,

$$\text{cobcat}(M^m \times N^n, f \times g) \le \text{cobcat}(N^n, g) - 1.$$

PROOF : Let cobcat $(M^m, f) = m_0$ and cobcat $(N^n, g) = n_0$. Let k be any integer such that $0 \le k \le m + n$. Let $i_1 + ... + i_p = k$ and $j_1 + ... + j_q = m + n - k$ be any two partitions such that $p + q \ge m_0 + n_0 - 1$. Now, consider the number

$$\langle W_{i_1} \dots W_{i_p} (f \times g)^* (z_{j_1} \dots z_{j_q}), \ [M^m \times N^n] \rangle$$

where $z_{j_a} \in H^{j_a}(X \times Y)$, for all $1 \le a \le q$. Note that this number can be expressed as sum of the numbers of the type

$$\langle (W_{r_1} \otimes W_{s_1}) \dots (W_{r_p} \otimes W_{s_p}) (f \times g)^* ((x_{k_1} \otimes y_{l_1}) .. (x_{k_q} \otimes y_{l_q})), \ [M^m \times N^n] \rangle$$

$$= \langle W_{r_1} \dots W_{r_p} f^* (x_{k_1} \dots x_{k_q}), [M^m] \rangle \langle W_{s_1} \dots W_{s_p} g^* (y_{l_1} \dots y_{l_q}), \ [N^n] \rangle$$

where for each $1 \le a \le p$ and for each $1 \le b \le q$, $r_a + s_a = i_a$, $k_b + l_b = j_b$, $x_{k_b} \in H^{k_b}(X)$ and $y_{l_b} \in H^{l_b}(Y)$. Now, r_a and s_a, and k_b and l_b can not be zero simultaneously, since $i_a > 0$ and $j_b > 0$. Suppose that the number of a's for which $r_a > 0$ is p' and the number of b's for which $k_b > 0$ is q'. Consequently, the number of a's for which $s_a > 0$ is atleast $(p - p')$, and the number of b's for which $l_b > 0$ is atleast $(q - q')$. Now, since $p + q \ge m_0 + n_0 - 1$, we have $p' + q' \ge m_0$, or $(p - p') + (q - q') \ge n_0$. Therefore, since cobcat $(M^m, f) = m_0$ and cobcat $(N^n, g) = n_0$, we have

$$\langle W_{r_1} \dots W_{r_p} f^* (x_{k_1} \dots x_{k_q}), \ [M^m] \rangle = 0$$

or,

$$\langle W_{s_1} \dots W_{s_p} g^* (y_{l_1} \dots y_{l_q}), \ [N^n] \rangle = 0.$$

Hence, it follows that the number

$$\langle W_{i_1} \dots W_{i_p} (f \times g)^* (z_{j_1} \dots z_{j_q}), \ [M^m \times N^n] \rangle = 0$$

This proves the proposition.

Remark 3.2 : If (M^n, f) is a singular manifold in a space X, and if $\varphi : X \to Y$ be any map from X to any space Y, then $(M^n, \varphi \circ f)$ is a singular manifold in Y with

$$\text{cobcat } (M^n, \varphi \circ f) \le \text{cobcat } (M^n, f).$$

Corollary 3.3 — If (M^m, f) and (N^n, g) ate two singular manifolds in an H-space X with $\mu_X : X \times X \to X$ giving the H-space structure, then

$$\text{cobcat } (M^m \times N^n, \ \mu_X \circ (f \times g)) \le \ \text{cobcat } (M^m, f) + \text{cobcat } (N^n, g) - 1.$$

PROOF : Follows immediately from Proposition 3.1 and Remark 3.2.

Corollary 3.4 — Let G be a compact abelian Lie group, and $(M^m; G)$ and $(N^n; G)$ be two G-manifolds. Then

$$G\text{-cobact } (M^m . N^n) \le G\text{-cobcat } (M^m) + G\text{-cobcat } (N^n) - 1.$$

PROOF : Let $f : (M^m . N^n)/G \to BG$, $f_M : M^m/G \to BG$ and $f_N : N^n/G \to BG$ be the classifying maps of the principal G-bundles $M^m . N^n \to (M^m . N^n)/G$, $M^m \to M^m/G$ and $N^n \to N^n/G$ respectively. Now, from the definition of . and the algebra isomorphism $N_*^G \cong N_*(BG)$, it follows that $((M^m . N^n)/G, f)$ is bordant to $(M^m/G \times \ N^n/G, \ \mu_{BG}$

$\circ (f_M \times f_N))$ in BG. Then

$$G\text{-cobcat } (M^m . N^n) = \text{cobcat } ((M^m . N^n)/G, f), \quad \text{by Remark 2.1}$$

$$= \text{cobcat } (M^m/G \times N^n/G\ ,\ \mu_{BG} \circ (f_M \times f_N)).$$

$$\leq \text{cobcat } (M^m/G, f_M) + \text{cobcat } (N^n/G, f_N) - 1, \quad \text{by Corollary 3.3}$$

$$= G\text{-cobcat } (M^m) + G\text{-cobcat } (N^n) - 1.$$

4. Calculation of Cobcat of Dold Manifolds

A Dold manifold $P(m, n)$ of dimension $m + 2n$ is the quotient $(S^m \times \mathbb{C} P^n)/\sim$, where $\sim$ is an equivalence relation given by $(x, [z]) \sim (-x, [\bar{z}])$. The ring structure of H^* $(P(m, n))$ can be described (cf. Dold[3]) as

$$H^* (P(m, n)) = \left[\frac{\mathbf{Z}_2 [c]}{c^{m+1} = 0} \right] \times \left[\frac{\mathbf{Z}_2 [d]}{d^{n+1} = 0} \right],$$

where c is the nonzero element of $H^1 (P(m, n)) = \mathbf{Z}_2$ and d is a suitable nonzero element of $H^2 (P(m, n)) = \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Further, the total Stiefel-Whitney class of $P(m, n)$ is given (cf. Dold) by $W = (1 + c)^m (1 + c + d)^{n+1}$.

Note that the index of nilpotency of $H^* (P(m, n))$ is $(m + n + 1)$. Hence, it follows that

$$\text{cobcat } (P(m, n)) \leq m + n + 1, \text{ for all } m, n.$$

Khare[4] has proved that $P(m, \text{odd})$ is a boundary for all m, and $P(m, 2r)$ is a boundary if and only if $m > 2r$ and 2^s divides $m - (2r + 1)$ for some $s > 0$ with $2^s > 2r$.

Therefore, the cobordism category of each of these Dold manifolds is 1 (Singh[5]).

Proposition 4.1 — If m, n are both even, then cobcat $(P(m, n)) = m + n + 1$.

Proof : The total Stiefel-Whitney class of $P(m, n)$ is given by

$$W = (1 + c)^m (1 + c + d)^{n+1}.$$

Since m, n are both even, $W_1 = c$ and $W_2 = d$ or $c^2 + d$.

Therefore,

$$W_1^m \, W_2^n = c^m \cdot d^n, \text{ since } c^{m+1} = 0$$
$$\neq 0.$$

Hence, it follows that cobcat $(P(m, n)) \geq m + n + 1$. But we always have cobcat $(P(m, n)) \leq m + n + 1$, and so cobcat $(P(m, n)) = m + n + 1$.

Proposition 4.2 — If m is odd and n is even with $m < n$, then cobcat $(P(m, n)) > n$.

PROOF : Note that here we have $W_1 = 0, W_2 = d$ or $c^2 + d$, and $W_3 = cd$. Therefore,

$$W_2^{n-m} W_3^m = d^{n-m}(cd)^m \quad \text{or} \quad (c^2+d)^{n-m}(cd)^m$$

$$= c^m d^n, \qquad \text{since } c^{m+1} = 0$$

$$\neq 0.$$

Thus, cobcat $(P(m, n)) > n - m + m = n$.

Proposition 4.3 — Let $m = 2^a - 1$ and $n = 2^b . s$, where $1 \leq a \leq b$ and $s \geq 1$. Then cobcat $(P(m, n)) = n + 1$.

PROOF : By (4.2), cobcat $(P(m, n)) \geq n + 1$. Now, $W = (1 + c)^m (1 + c + d)^{n+1}$

$$= (1 + (1 + c + c^2 + ... + c^{2^a - 1})\, d)\,(1 + d^{2^b})^s, \text{ as } c^{2^b} = 0.$$

Hence, it follows that all the nonzero W_i's are multiples of d. So, any product of Stiefel-Whitney classes of length more than n will be zero, as $d^{n+1} = 0$. This proves the proposition.

Note that this proposition gives the cobordism category of all Dold manifolds lying in the generating set of N_*.

Proposition 4.4 — Let $m = 2^a - 1$ and $n = 2^b . s$, where $a > b > 1$ and s is odd. Then cobcat $(P(m, n)) = 2^{a-b} + n$.

PROOF : Here we have

$$W = (1 + (1 + c + c^2 + ... c^{2^a - 1})d)\,(1 + c^{2^b} + d^{2^b})^s.$$

So, $W_1 = 0, W_2 = d, W_3 = cd$ and $W_{2^b} = c^{2^b} + c^{2^b - 2} . d$, since s is odd. Note that if, for some i, W_i has a term like c^i then we must have $i \geq 2^b$. Now, one can see that

$$W_{2^b}^{2^{a-b}-1}\, W_3^{2^b - 1}\, W_2^{2^b . s - 2^b + 1} \neq 0.$$

So, cobcat $(P(m, n)) \geq 2^{a-b} + 2^b . s$.

Now, suppose that, in dimension $m + 2n$, there is a nonzero product of Stiefel-Whitney classes whose length is more than $2^{a-b} + 2^b . s - 1$. Further, suppose in that product the number of W_i's which are multiples of d is p, and the number of W_i's which are not multiples of d is q. By assumption,

$$p + q > 2^{a-b} + 2^b . s - 1 \qquad ... (1)$$

Note that the product of q Stiefel-Whitney classes which are not multiples of d looks like

$$c^i + (\text{terms which are multiples of } d),$$

where one can see that $i \geq 2^b . q$. So, the whole product of $(p + q)$ Stiefel-Whitney classes will have a term which is a multiple of $c^i . d^p, i \geq 2^b . q$, and the rest of the terms will be multiples of d^{p+1}. Thus the whole product will be zero if $p \geq 2^b . s$, noting that if $p = 2^b . s$ then, by (1), $q \geq 2^{a-b}$. Hence we assume that $p < 2^b . s$. Then, by (1),

$$q \geq 2^{a-b} + 1 \qquad \text{... (2)}$$

Now, since the whole product is considered in dimension $m + 2n$, one observes that

$$m + 2n \geq 2^b . q + 2p$$

or,

$$2^a - 1 + 2^{b+1} . s \geq 2^b . q + 2p \qquad \text{... (3)}$$

Again, by (2),

$$2^b . q - 2q \geq 2^a - 2^{a-b+1} + 2^b - 2.$$

So,

$$\begin{aligned} 2^b . q + 2p &\geq 2p + 2q + 2^a - 2^{a-b+1} + 2^b - 2 \\ &\geq 2p + 2q + 2^a - 2^{a-b+1} + 1, \qquad \text{as} \quad b > 1 \\ &> 2^a + 2^{b+1} . s - 1, \text{ by } (1) \end{aligned}$$

which contradicts (3). Hence, in dimension $m + 2n$, there does not exist any nonzero product of Stiefel-Whitney classes whose length is more than $2^{a-b} + 2^b . s - 1$. This proves the proposition.

Efforts are being made to tackle the remaining classes of Dold manifolds; however, due to the lack of neatness and completeness of the calculations involved there in, they are not included here. We hope to put them in our forthcoming works.

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