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OSCILLATORY AND NONOSCILLATORY BEHAVIOUR OF SOLUTONS OF AN EQUATION ALTERNATELY OF RETARDED AND ADVANCED TYPE

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Sufficient conditions have been obtained for oscillation and nonoscillation of solutions of a nonlinear differential equation which is alternately of retarded and advanced type.

§ 1. During last few years many people have worked on differnetial equations with piece-wise constant arguments. These equations have the structure of continuous dynamical systems within intervals of certain length and are closely related to impulse and loaded equations. Further, they are similar in structure to those found in certain "Sequential continuous" models of disease dynamics as treated by Busenberg and Cooke[2].

Cooke and Wiener[4] have considered

$$x'(t) = a(t)\,x(t) + a_0(t)\,x(2[(t+1)/2]),\quad x(0) = c_0, \qquad \dots (1)$$

where a_0 and $a \in C(0, \infty), R)$ and [.] is the greatest integer function The argument deviation

$$\tau(t) = t - 2[(t+1)/2]$$

is negative for $2n - 1 \le t < 2n$ and positive for $2n < t < 2n + 1$, where n is a positive integer. Clearly, eqn. (1) is of advanced type on $[2n - 1, 2n)$ and of retarded type on $(2n, 2n + 1)$. They have obtained a result concerning existence of a unique solution of (1) on $[0, \infty)$. Sufficient conditons have been obtained for oscillation of solutions of (1). When $a_0(t)$ and $a_1(t)$ in (1) are constants, Cooke and Wiener[4] have obtined results concerning existence of a unique solution of (1) on $[0, \infty)$, backward continuation on $(-\infty, 0]$ and necessary and sufficient conditions for nonoscillation of (1).

In this paper we consider differential equations with piece-wise constant

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arguments of the type

$$x'(t) + p(t)\,x(t) + q(t)\,f(x\,(2\,[(t+1)/2])) = 0, \; x(0) = c_0, \qquad \text{... (2)}$$

where p and $q \in C([0, \infty), R)$ and $f \in C(R, R)$, R being the real-line. Our results on oscillation and nonoscillation for (2) generalize those of Cooke and Wiener[4] for (1).

Definition 1 — A solution of (2) on $[0, \infty)$ is a function $x(t)$ that satisfies the conditions : (i) $x(t)$ is continuous on $[0, \infty)$, (ii) The derivative $x'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $t = 2n - 1 (n = 1, 2, ...)$, where one-sided derivatives exist, (iii) eqn. (2) is satisfied on each interval $2n - 1 \leq t < 2n + 1$ and (iv) $x(0) = c_0$ and $x(t)$ is nontrivial in any neighburhood of infinity.

Definition 2 — A solution $x(t)$ of (2) on $[0, \infty)$ is said to be oscillatory if it has infinitely large zeros; otherwise, it is said to be nonoscillaory. Equation (2) is said to be oscillatory if all of its solutions are oscillatory. It is said to be nonoscillatory if all of its solutions are nonoscillatory.

The existence problem for eqn. (2) has been considered in Section 2. Oscillatory and nonoscillatory behaviour of solutions of (2) have been dealt with in Section 3. The last section is concerned with the oscillation and nonoscillation of a forced differential equation.

§ 2. We begin with the following proposition.

Proposition 1 — Suppose that $yf(y) > 0$ for $y \neq 0$. If $x(t)$ is a solution of (2) on $[0, \infty)$ such that $x(2n) = 0$ for $n = m, m + 1, m + 2, ...,$ where $m \geq 0$ is an integer, then $x(t) = 0$ for $t \in [2m, \infty)$.

PROOF : For $t \in [2m, 2m + 1)$, (2) reduces to

$$(x(t) \exp (\int_{2m}^{t} p(s)\, ds))' = 0$$

So $x(t) = 0$ for $t \in [2m, 2m + 1)$. From the continuity of $x(t)$ it follows that $x(t) = 0$ for $t \in [2m, 2m + 1]$. For $t \in [2m + 1, 2m + 2)$, we get

$$(x(t) \exp (\int_{2m+1}^{t} p(s)\, ds))' = 0$$

Thus $x(t) = 0$ for $t \in [2m + 1, 2m + 2]$. Proceeding in this way one can show that $x(t) = 0$ for $t \geq 2m$.

Hence the proposition is proved.

In what follows we discuss the existence problems concerning (1) and (2). For convenience we state a theorem due to Cooke and Wiener[4] (Theorem 6) in the following.

Theorem 2 — Equation (1) admits a unique solution on $[0, \infty)$ if

$$(\mathrm{H}_1) \quad \int_{2n-1}^{2n} u^{-1}(t)\, a_0(t)\, dt \neq u^{-1}(2n), \;\; n = 1, 2, ...,$$

where u^{-1} is the reciprocal of u and $u(t) = \exp (\int_0^t a(s)\; ds)$.

Remark 1 : In the following, Proposition 1 is used to show that the assumption

(H_1) above due to Cooke and Wiener is not enough to obtain a unique solution of (1) which is nontrivial in any neighbourhood of infinity. One needs a stronger assumption (H_2) (see below) for this purpose.

If $x(t)$ is a solution of (1) on $[0, \infty)$ with $x(0) = c_0$, then

$$x(t) = c_0 u(t) \left(1 + \int_0^t a_0(s) u^{-1}(s)\, ds\right), \quad 0 \le t \le 1,$$

$$u^{-1}(t) x(t) = u^{-1}(1) x(1) + x(2) \int_1^t a_0(s) u^{-1}(s)\, ds, \quad 1 \le t \le 2,$$

$$u^{-1}(t) x(t) = u^{-1}(2) x(2) + x(2) \int_2^t a_0(s) u^{-1}(s)\, ds, \quad 2 \le t \le 3,$$

$$u^{-1}(t) x(t) = u^{-1}(3) x(3) + x(4) \int_3^t a_0(s) u^{-1}(s)\, ds, \quad 3 \le t \le 4,$$

and so on. It (H_1) holds, then from Proposition 1 it follows that $x(t) = 0$ for $t \ge 0$ if $c_0 = 0$. Suppose that $c_0 \ne 0$. If (H_1) holds and

$$1 + \int_0^1 a_0(s) u^{-1}(s)\, ds = 0,$$

then $x(t) = 0$ for $t \ge 1$. On the other hand, if

$$1 + \int_0^1 a_0(s) u^{-1}(s)\, ds \ne 0$$

but $u^{-1}(2) + \int_2^3 a_0(s) u^{-1}(s)\, ds = 0$, then $x(t) = 0$ for $t \ge 3$ and so on. Hence, in order to get a solution of (1) which is nontrivial in any neighbourhood of infinity, we should assume

$$(H_2) \quad u^{-1}(2n) + \int_{2n}^{2n+1} a_0(s) u^{-1}(s)\, ds \ne 0, \; n = 0, 1, 2, \ldots,$$

instead of (H_1) in Theorem 2. It is interesting to note that (H_2) implies (H_1). Indeed, if (H_2) holds for $n = 0$, then $x(1) \ne 0$. This in turn implies that $x(2) \ne 0$ and (H_1) holds for $n = 1$. This process continues and hence our assertion holds.

The above remark holds for (2) if f satisfies the condition $yf(y) > 0$ for $y \ne 0$. Hence we have the following existence theorem for (2).

Theorem 3 — Consider (2) with $x(0) = c_0 \ne 0$. Suppose that $yf(y) > 0$ for $y \ne 0$.

$$(H_3) \; \lambda v^{-1}(2n) \ne f(\lambda) \int_{2n}^{2n+1} q(t) v^{-1}(t)\, dt, \text{ for } \lambda \ne 0, \text{ n} = 0, 1, 2, 3, \ldots, \text{ and}$$

$$c_{2n} v^{-1}(2n) + f(c_{2n}) \int_{2n-1}^{2n} q(t) v^{-1}(t)\, dt = c_{2n-1} v^{-1}(2n-1)$$

admits a solution c_{2n}, $n = 1, 2, \ldots$, where $v(t) = \exp\left(-\int_0^t p(s)\,ds\right)$, and v^{-1} is the reciprocal of v. Then (2) admits a unique solution on $[0, \infty)$.

Remark 2 : (i) If $f(x) = x$, then (H_3) reduces to (H_2).

(ii) If $q(t) \leq 0$, then ((H_3) is satisfied trivially.

For backward continuation of solutions of (2) we need the following proposition the proof of which is similar to that of proposition 1 and hence is omitted. We assume that p and q in (2) are real valued continuous functions on $(-\infty, 0]$.

Proposition 4 — Let $yf(y) > 0$ for $y \neq 0$. If $x(t)$ is a solution of (2) on $(-\infty, 0]$ such that $x(-2n) = 0$, where $n = m, m+1, \ldots$, and $m \geq 0$ is an integer, then $x(t) = 0$ for $t \in (-\infty, -2m]$.

Theorem 5 — Consider (2) with $x(0) = c_0 \neq 0$. Suppose that $yf(y) > 0$ for $y \neq 0$,

$$\lambda u(-2n) + f(\lambda) \int_{-2n-1}^{-2n} q(t)\, u(t)\, dt \neq 0$$

for every $\lambda \neq 0$, $n = 0, 1, 2, \ldots$, where $u(t) = \exp\left(-\int_t^0 p(s)\,ds\right)$,

and

$$c_{2n-1}\, u(1-2n) = c_{2n}\, u(-2n) - f(c_{2n}) \int_{-2n}^{-2n+1} q(s)\, u(s)\, ds,$$

$n = 1, 2, \ldots,$ is solvable for c_{2n} provided that c_{2n-1} is known. Then (2) admits a unique solution on $(-\infty, 0]$.

The proof is similar to that of Theorem 3 and hence is omitted.

§ 3. In this section we obtain sufficient conditions for oscillation and nonoscillation of (2). All the conditions of Theorem 3 are assumed to be satisfied and hence the existence of a unique solution of (2) on $[0, \infty)$ follows.

Theorem 6 — Let $q(t) > 0$. If $0 < f(y)/y \leq M$ for $y \neq 0$ and

$$(H_4) \quad \limsup_{m \to \infty} \int_{2m}^{2m+1} q(t) \exp\left(\int_{2m}^{t} p(s)\,ds\right) dt < 1/M,$$

where $M > 0$ is a constant, then (2) is nonoscillatory.

PROOF : From the given hypothesis it follows that there exists an integer $N > 0$ such that

$$M \int_{2m}^{2m+1} q(t) \exp\left(\int_{2m}^{t} p(s)\,ds\right) dt < 1$$

for $m \geq N$. It is clear from the Proposition 1 that there exists an integer $n \geq N$ such that $x(2n) \neq 0$, where $x(t)$ is a solution of (2) on $[0, \infty)$. Suppose that $x(2n) > 0$. For $2n \leq t < 2n+1$, Equation (2) reduces to

$$(x(t) \exp(\int_{2n}^{t} p(s)\,ds))' + f(x(2n))\,q(t) \exp(\int_{2n}^{t} p(s)\,ds) = 0.$$

Integrating this from $2n$ to t, we get

$$x(t) \exp(\int_{2n}^{t} p(s)\,ds)$$

$$= x(2n) \left\{ 1 - (f(x(2n))/x(2n)) \int_{2n}^{t} q(\theta) \exp(\int_{2n}^{\theta} p(s)\,ds)\,d\theta \right\}$$

$$\geq x(2n) \left\{ 1 - (f(x(2n))/x(2n)) \int_{2n}^{2n+1} q(\theta) \exp(\int_{2n}^{\theta} p(s)\,ds)\,d\theta \right\} > 0.$$

From the continuity of $x(t)$ it follows that $x(2n+1) > 0$. So $x(t) > 0$ for $2n \leq t \leq 2n+1$. For $2n+1 \leq t \leq 2n+2$, (2) takes the form

$$(x(t) \exp(\int_{2n+1}^{t} p(s)\,ds))' + f(x(2n+2))\,q(t) \exp(\int_{2n+1}^{t} p(s)\,ds) = 0. \quad \ldots (3)$$

Integrating from $2n+1$ to $2n+2$, we obtain

$$x(2n+1) = x(2n+2) \exp(\int_{2n+1}^{2n+2} p(s)\,ds)$$

$$+ f(x(2n+2)) \int_{2n+1}^{2n+2} q(t) \exp(\int_{2n+1}^{t} p(s)\,ds)\,dt. \quad \ldots (4)$$

From the given condition it is clear that $f(0) = 0$. As $x(2n+2) = 0$ implies that $x(2n+1) = 0$, $x(2n+2) \neq 0$. So

$$x(2n+1) = x(2n+2)$$

$$\times \left\{ \exp(\int_{2n+1}^{2n+2} p(s)\,ds) + (f(x(2n+2))/x(2n+2)) \int_{2n+1}^{2n+2} q(t) \exp(\int_{2n}^{t} p(s)\,ds)\,dt \right\}.$$

This in turn implies that $x(2n+2) > 0$. We claim that $x(t) > 0$ for $t \in (2n+1, 2n+2)$. If not, there exists a point $t_1 \in (2n+1, 2n+2)$ such that $x(t_1) = 0 = x'(t_1)$ and $x(t) > 0$ for $2n+1 \leq t < t_1$ and $t_1 < t \leq 2n+2$ or there exist two points t_1 and t_2 $(2n+1 < t_1 < t_2 < 2n+2)$ such that $x(t_1) = 0 = x(t_2)$ and $x(t) < 0$ for $t_1 < t < t_2$. In the first case, from the given equation (2), we obtain $0 < q(t_1) f(x(2n+2)) = 0$, a contradiction. In the second case, we integrate (3) from t_1 to t_2 to get

$$0 < f(x(2n+2)) \int_{t_1}^{t_2} q(t) \exp\left(\int_{2n+1}^{t} p(s)\, ds\right) dt = 0,$$

a contradiction again. Hence our assertion holds, that is, $x(t) > 0$ for $2n + 1 \le t \le 2n+2$. Continuing this process we get $x(t) > 0$ for $t \ge 2n$.

If $x(2n) < 0$, then continuing as above we get $x(t) < 0$ for $t \ge 2n$. Hence the theorem is completed.

Remark 3 : Define $f : R \to R$ by

$$f(y) = \begin{cases} y^{1/3}, & y \le -1 \\ y^3, & -1 \le y \le 1 \\ y^{1/3}, & y \ge 1 \end{cases}$$

Clearly, f satisfies the conditions of Theorem 6.

Theorem 7 — Suppose that $q(t) < 0$. If $0 < f(y)/y \le M$ for $y \ne 0$ and

$$(H_5) \quad \liminf_{m \to \infty} \int_{2m+1}^{2m+2} q(t) \exp\left(-\int_{t}^{2m+2} p(s)\, ds\right) dt > -1/M,$$

where $M > 0$ is a constant, then (2) is nonoscillatory.

PROOF : It is clear from the given condition that there exists an integer $N > 0$ such that $m \ge N$ implies

$$M \int_{2m+1}^{2m+2} q(t) \exp\left(-\int_{t}^{2m+2} p(s)\, ds\right) dt > -1.$$

Further, there exists an integer $n \ge N$ such that $x(2n) \ne 0$. Let $x(2n) > 0$. The case $x(2n) < 0$ may be dealt with similarly. For $2n \le t < 2n+1$, we get

$$x(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right)$$

$$= x(2n) \left\{ 1 - (f(x(2n))/x(2n)) \int_{2n}^{t} q(\theta) \exp\left(\int_{2n}^{\theta} p(s)\, ds\right) d\theta \right\}.$$

Since $q(t) < 0$, then $x(t) > 0$ for $t \in [2n, 2n+1]$. Next, for $2n + 1 \le t \le 2n+2$, we have (4). Clearly, $x(2n + 2) \ne 0$. Now

$$x(2n+1) = x(2n+2) \exp\left(\int_{2n+1}^{2n+2} p(s)\, ds\right) \times$$

$$\left\{ 1 + (f(x(2n+2))/x(2n+2)) \int_{2n+1}^{2n+2} q(t) \exp\left(-\int_{t}^{2n+2} p(s)\, ds\right) dt \right\}.$$

Hence $x(2n + 1) > 0$ and (H_5) imply that $x(2n + 2) > 0$. Proceeding as in Theorem 6, we may show that $x(t) > 0$ for $t \in [2n+1, 2n+2]$. Repeating the process we get $x(t) > 0$ for $t \geq 2n$.

This completes the proof of the theorem.

Remark 4 : Conditions (H_4) and (H_5) are sharpe. If $f(y) = y$ and $p(t)$ and $q(t)$ are constants p and q respectively, then (H_4) is reduced to

$$\frac{pe^p}{1-e^p} < 0 < q < \frac{p}{e^p - 1}$$

and (H_5) is reduced to

$$\frac{pe^p}{1-e^p} < q < 0 < \frac{p}{e^p - 1}.$$

These bounds are same as those obtained by Cooke and Wiener[4] (see Theorem 5).

Remark 5 : Neither Theorem 6 nor 7 is valid if $f(y)$ is super-linear or sublinear. However, the following are theorems if $f(y)$ is super-linear.

Theorem 8 — Suppose that $q(t) < 0$, $yf(y) > 0$ for $y \neq 0$ and $f(y)/y \leq M$ for $y \neq 0$ and $|y| \leq K$, where $M = M(K) > 0$ is a constant. If

$$\lim_{m \to \infty} \int_{2m}^{2m+1} q(t) \exp\left(\int_{2m}^{t} p(s)\, ds \right) dt = 0,$$

then all bounded solutions of (2) are nonoscillatory.

Theorem 9 — Suppose that $q(t) < 0$, $yf(y) > 0$ for $y \neq 0$ and $f(y)/y \leq M$ for $y \neq 0$ and $|y| \leq K$, where $M = M(K) > 0$ is a constant. If

$$\lim_{m \to \infty} \int_{2m+1}^{2m+2} q(t) \exp\left(-\int_{t}^{2m+2} p(s)\, ds \right) dt = 0,$$

then all bounded solutions of (2) are nonoscillatory.

The proof of Theorem 8 or 9 is similar to that of Theorem 6 and hence is omitted.

Theorem 10 — Let $q(t) \geq 0$ and $f(y)/y \geq M$ for $y \neq 0$, where $M > 0$ is a constant. If

$$\limsup_{m \to \infty} \int_{2m}^{2m+1} q(t) \exp\left(\int_{2m}^{t} p(s)\, ds \right) dt > 1/M,$$

then (2) is oscillatory.

PROOF : Let $x(t)$ be a solution of (2) on $[0, \infty)$. If possible, let $x(t) > 0$ for $t \geq 2n$, where n is a sufficiently large positive integer. For $2n \leq t \leq 2n+1$, (2) reduces to

$$(x(t) \exp (\int_{2n}^{t} p(s)\, ds))' + f(x(2n))\, q(t) \exp (\int_{2n}^{t} p(s)\, ds) = 0. \qquad \dots (5)$$

Integrating from $2n$ to $2n + 1$, we get

$$x(2n+1) \exp (\int_{2n}^{2n+1} p(s)\, ds)$$

$$= x(2n) \left\{ 1 - (f(x(2n))/x(2n)) \int_{2n}^{2n+1} q(t) \exp (\int_{2n}^{t} p(s)\, ds)\, dt \right\}$$

$$\leq x(2n) \left\{ 1 - M \int_{2n}^{2n+1} q(t) \exp (\int_{2n}^{t} p(s)\, ds)\, dt \right\}$$

$x(2n + 1) > 0$ implies that

$$M \int_{2n}^{2n+1} q(t) \exp (\int_{2n}^{t} p(s)\, ds)\, dt < 1,$$

and hence

$$\limsup_{n \to \infty} \int_{2n}^{2n+1} q(t) \exp (\int_{2n}^{t} p(s)\, ds)\, dt \leq 1/M,$$

a contradicton.

If $x(t) < 0$ for $t \geq 2n$, then, for $2n \leq t < 2n + 1$, we get from (5)

$$x(2n+1) \exp (\int_{2n}^{2n+1} p(s)\, ds) \geq x(2n) \left\{ 1 - M \int_{2n}^{2n+1} q(t) \exp (\int_{2n}^{t} p(s)\, ds) dt \right\}$$

Clearly, $x(2n + 1) < 0$ leads to the same contradiction.

Thus the theorem is proved.

Theorem 11 — Suppose that $q(t) \leq 0$ and $f(y)/y \geq M$ for $y \neq 0$, where $M > 0$ is a constant. If

$$\lim_{m \to \infty} \inf \int_{2m-1}^{2m} q(t) \exp (\int_{2m}^{t} p(s)\, ds)\, dt < -1/M,$$

then (2) is oscillatory.

The proof of this theorem is same as that of Theorem 10 and hence is omitted.

Remark 6 : (i) Theorems 10 and 11 generalize, respectively, Theorems 9 and 12 in Cooke and Wiener[4] (ii) The following function $f : R \to R$ satisfies the conditons of Theorem 10 or 11.

$$f(y) = \begin{cases} y^3, & y \le -1 \\ y^{1/3}, & -1 \le y \le 1. \\ y^3, & y \ge 1 \end{cases}$$

Theorem 12 — Suppose that $q(t) \ge 0$, $\left| \int_0^\infty p(t)\,dt \right| < \infty$, $yf(t) > 0$ for $y \ne 0$ and $\lim_{y \to 0} (y/f(y)) = M < \infty$. If

$$\limsup_{m \to \infty} \int_{2m}^{2m+1} q(t) \exp\left(\int_\sigma^t p(s)\,ds\right) dt > M \exp\left(\int_\sigma^\infty p(t)\,dt\right).,$$

for every $\sigma \ge 0$, then (2) is oscillatory.

PROOF : Let $x(t)$ be a solution of (2) on $[0, \infty)$ such that $x(t) > 0$ for $t \ge 2n$, where n is a sufficiently large positive integer. The case when $x(t) < 0$ for $t \ge 2n$ may similarly be dealt with. Setting.

$$z(t) = x(t) \exp\left(\int_{2n}^t p(s)\,ds\right),$$

We obtain from (2) that $z'(t) \le 0$ for $t \ge 2n$. So $\lim_{t \to \infty} z(t)$ exists. Since $\left| \int_0^\infty p(t)\,dt \right| < \infty$, then $\lim_{t \to \infty} x(t)$ exists. Suppose that $\lim_{t \to \infty} x(t) = \alpha$. Thus $\alpha \ge 0$. Let $\alpha > 0$.

From the given hypothesis it follows that there exists a sequence $\langle m_j \rangle$ and an $\varepsilon > 0$ such that $m_j \to \infty$ and

$$\int_{2m_j}^{2m_j+1} q(t) \exp\left(\int_{2n}^t p(s)ds\right) dt > M \exp\left(\int_{2n}^\infty p(t)\,dt\right) + \varepsilon$$

For $0 < \varepsilon_1 < f(\alpha)$, there exists a positive integer $N > n$ such that $f(x(2m_j)) > f(\alpha) - \varepsilon_1$ for $m_j \ge N$. Now integrating (2) from $2N$ to $2N + 2r + 1$, where r is a positive integer, we obtain

$$z(2N + 2r + 1) - z(2N)$$

$$= -\int_{2N}^{2N+2r+1} q(t) \exp\left(\int_{2n}^t p(s)\,ds\right) f(x(2[(t+1)/2]))dt$$

$$\le -f(x(2N)) \int_{2N}^{2N+1} q(t) \exp\left(\int_{2n}^t p(s)\,ds\right) dt$$

$$-f(x(2N+2)) \int_{2N+2}^{2N+3} q(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right) dt - \ldots -$$

$$- f(x(2N+2r)) \int_{2N+2r}^{2N+2r+1} q(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right) dt$$

$$< -(r+1)(f(\alpha) - \varepsilon_1)\left(M \exp\left(\int_{2n}^{\infty} p(t)\, dt\right) + \varepsilon\right).$$

This in turn implies that $z(t) < 0$ for large t, a contradiction.

If $\alpha = 0$, then integrating (2) from $2m_j$ to $2m_j + 1$, $m_j \geq n$, we get

$$-z(2m_j) < z(2m_j+1) - z(2m_j)$$

$$< -f(x(2m_j)) \int_{2m_j}^{2m_j+1} q(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right) dt.$$

Thus

$$\int_{2m_j}^{2m_j+1} q(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right) dt < (x(2m_j)/f(x(2m_j))) \times \exp\left(\int_{2n}^{2m_j} p(t)\, dt\right)$$

So

$$\limsup_{m_j \to \infty} \int_{2m_j}^{2m_j+1} q(t) \exp\left(\int_{2n}^{t} p(s)\, ds\right) dt \leq M \exp\left(\int_{2n}^{\infty} p(t)\, dt\right),$$

a contradiction.

This completes the proof of the theorem.

Theorem 13 — Suppose that $q(t) \leq 0$, $\left|\int_{0}^{\infty} p(t)\, dt\right| < \infty$ and $yf(y) > 0$ for $y \neq 0$. If

$$\liminf_{m \to \infty} \int_{2m}^{2m+1} q(t) \exp\left(\int_{\sigma}^{t} p(s)\, ds\right) dt < -M \exp\left(\int_{\sigma}^{\infty} p(t)\, dt\right),$$

$M > 0$ is a constant, then all bounded solutions of (2) are oscillatory.

The proof of this theorem is same as that of Theorem 12 and hence is omitted.

4. In this section we obtain sufficient conditions for oscillation and nonoscillation of forced differential equation

$$x'(t) + p(t)x(t) + q(t) f(x(2[(t+1)/2])) = h(t), x(0) = c_0, \qquad \ldots (6)$$

where p, q and f are same as in (2) and $h \in C([0, \infty), R)$. It is assumed that (6) admits a solution on $[0, \infty)$.

In the following we obtain some oscillation and nonoscillation results for

$$x'(t) + q(t) f(x(2[(t+1)/2])) = h(t), \; x(0) = c_0. \qquad \dots (7)$$

This analysis may easily be extended to (6).

Theorem 14 — Let $q(t) \geq 0$, $yf(y) > 0$ for $y \neq 0$ and f be monotonic increasing. Suppose that there exists a function $H \in C^1([0, \infty), R)$ such that $H'(t) = h(t)$. If $\Gamma_1 = \infty$ and $\Gamma_2 = -\infty$, where

$$\Gamma_1 = \sum_{m=0}^{\infty} f(H_+(2m)) \int_{2m}^{2m+1} q(t)\,dt,$$

$$\Gamma_2 = \sum_{m=0}^{\infty} f(-H_-(2m)) \int_{2m}^{2m+1} q(t)\,dt,$$

$$H_+(t) = \max\{h(t), 0\} \text{ and } H_-(t) = \max\{-H(t), 0\},$$

then (7) is oscillatory.

PROOF : Let $x(t)$ be a solution of (7) on $[0, \infty)$. If possible let $x(t) > 0$ for $t \geq 2n$, where $n > 0$ is a succifiently large integer. Setting $z(t) = x(t) - H(t)$, we obtain from (7) that

$$z'(t) = -q(t) f(x(2[(t+1)/2])) \leq 0 \qquad \dots (8)$$

for $t \geq 2n$. So there exists an integer $N \geq n$ such that $z(t) > 0$ or < 0 for $t \geq 2N$. But $z(t) < 0$ for $t \geq 2N$ implies that $H(t) > 0$ for $t \geq 2N$. Thus $\Gamma_2 > -\infty$, a contradiction.

Now $z(t) > 0$ for $t \geq 2N$ implies that $H(t) < x(t)$, that is, $H_+(t) < x(t)$, for $t \geq 2N$, that is,

$$f(H_+(2[(t+1)/2])) \leq f(x(2[(t+1)/2])). \qquad \dots (9)$$

Integrating (8) from $2N$ to $t(2N < t)$, we get

$$\int_{2N}^{t} q(s) f(x(2[(s+1)/2]))\,ds = z(2N) - z(t) < z(2N).$$

This in turn implies that

$$\int_{2N}^{\infty} q(t) f(x(2[(t+1)/2]))\,dt < \infty.$$

Thus from (9) we obtain

$$\int_{2N}^{\infty} q(t) f(H_+(2[(t+1)/2]))\,dt < \infty,$$

that is,

$$\sum_{m=N}^{\infty} f(H_+(2m)) \int_{2m}^{2m+1} q(t)\, dt < \infty.$$

Thus $\Gamma_1 < \infty$, a contradiction.

Next let $x(t) < 0$ for $t \geq 2n$. So $z'(t) \geq 0$ for $t \geq 2n$. Thus there exists an integer $N \geq n$ such that $z(t) > 0$ or < 0 for $t \geq 2N$. However, $z(t) > 0$ for $t \geq 2N$. implies that $\Gamma_1 < \infty$, a contradictcion. Therefore, $z(t) < 0$ for $t \geq 2N$. This in turn implies that $x(t) < -H_-(t)$. Proceeding as above we obtain that $\Gamma_2 > -\infty$, a contradiction.

Hence the theorem is proved.

Theorem 15 — Let $q(t) \leq 0$, $yf(y) > 0$ for $y \neq 0$ and f be monotonio increasing. Suppose that there exists a bounded function $H \in C^1([0, \infty), R)$ such that $H'(t) = h(t)$. If $\Gamma_1 = -\infty$ and $\Gamma_2 = \infty$, where Γ_1 and Γ_2 are same as in Theorem 14, then bounded solutions of (7) oscillate.

The proof is similar to that of Theorem 14 and hence is omitted.

Theorem 16 — Let $h(t) \geq 0$. If $\lim_{t \to \infty} (h(t)/|q(t)|) = \infty$, whenever it is defined, then all bounded solutions of (7) are nonoscillatory.

PROOF : Let $x(t)$ be a bounded solution of (7) on $[0, \infty)$ such that $|x(t)| \leq M$ for $t \geq 0$. There exists a constant $K > 0$ such that $|f(y)| \leq K$ for $y \in [-M, M]$. From the given hypothesis it is clear that $h(t) \geq K|q(t)|$ for $t \geq T$, where $T > 0$ is sufficiently large.

If $x(t)$ is oscillatory, then there exists a sequence $\langle t_m \rangle$ such that $t_m \to \infty$ as $m \to \infty$ and $x(t_m) = 0$. Choose m sufficiently large so that $t_m > T$. Integrating (7) from t_m to t_{m+1}, we get

$$0 \geq \int_{t_m}^{t_{m+1}} (h(t) - K|q(t)|)\, dt > 0,$$

a contradiction.

This completes the proof of the theorem.

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