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STABILITY OF FLOW OF A VISCOUS INCOMPRESSIBLE FLUID BETWEEN CONCENTRIC ROTATING CYLINDERS WITH RADIAL TEMPERATURE GRADIENT

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A trigonometric series method is used to solve the eigenvalue problem of the stability of a viscous incompressible fluid between two concentric cylinder with either the inner one rotating or both are corotating. It is assumed that the gap is narrow and a radial temperature gradient exists. The numerical values of a_c (the critical wave number) and Ta_c (the critical Taylor number) are computed and these agree very well with earlier results computed numerically. The flow remains more stable when the temperature of the outer cylinder is maintained at a level lower than that of the inner cylinder.

NOMENCLATURE

a	=	dimensionless wave number;
A	=	see eqn (4);
$A_1(m), A_2(m)$	=	constants of integration;
$B_1(m), B_2(m)$	=	constants of integration;
d	=	difference between two radii of the cylinders;
D	=	constant coefficient (eqn.) (5));
N	=	Ra/Ta;
r	=	distance from the axis;

R_1, R_2	=	radii of the inner and outer cylinders respectively;
Ra	=	Rayleigh number;
T_1, T_2	=	temperatures of the inner and outer cylinders respectively;
Ta	=	Taylor number;
$u,\ v$	=	dimension less components of velocity in the r, θ directions;
X_{mn}, Y_{mn}	=	see eqn. (16).

Greek symbols

α'	=	coefficient of volume expansion;
α_m	=	see equations (11)-(14);
β	=	coefficient of thermal expansion;
β_m, γ_m	=	see equations (11)-(14);
Δ	=	$\sin h^2 a - a^2$;
k	=	thermal conductivity;
λ	=	a/d;
μ	=	Ω_2/Ω_1;
ν	=	kinematic viscosity;
Ω_2/Ω_1	=	angular velocities of inner and outer cylinders.

1. INTRODUCTION

Stability of the flow of a viscous incompressible fluid in a narrow annular region of two concentric cylinders, with inner one rotating, was first studied by Taylor[7], both theoretically and experimentally. He noticed that the formation of vortices depends upon the speed of the inner rotating cylinder. Thus the critical values of Taylor number Ta_c and the critical wave number a_c were determined by Taylor. Chandrasekhar[2] gave an alternate and elegant method of solution which consisted of a trigonometric series solution to determine the eigenvalues. Exact numerical method for determining the eigenvalues Ta_c, a_c of Taylor stability problem was first given by Harris and Reid[3] for a narrow-gap Taylor problem. In these investigations, the concentric cylinders were assumed to be at the same temperature and hence the existence of radial temperature gradient was not considered in all these investigations. So Walowit *et al.*[8] noticed the importance of the effects of a radial temperature gradient and systematically analysed the stability of Taylor problem for both wide-gap and narrow-gap cases. The Galerkin method was employed for deriving the eigenvalues Ta_c and a_c. Recently, Soundalgekar *et al.*[4], presented an exact solution to the Taylor stability problem in case of a narrow-gap and also considered the effects of the radial temperature gradient in the presence of a rotating inner cylinder.

The numerical results for Ta_c and a_c were compared with those of Walowit *et al.*[8] and Harris and Reid[3] and the agreement was good. The stability of the narrow-gap Taylor problem, in the presence of a radial temperature gradient and both the cylinders rotating was also studied by Takhar *et al.*[5] where again an exact numerical solution was presented. The results agreed well with those of Harris and Reid[3]. Many papers are published on this topic and these are referred in Soundalgekar *et al.*[4]

It is now proposed to study the narrow-gap Taylor-stability problem on taking into account the radial temperature gradient and either inner or both the cylinders rotating in the same direction. The problem is solved by Chandrasekhar's trigonometric series method and the results are compared with those obtained by Takhar *et al.*[6]. This also worked as a check on the numerical method employed in Takhar *et al.*[6]. In section 2, the mathematical analysis is presented and in section 3, the conclusions are set out.

2. Mathematical Analysis

Consider the flow of an incompressible viscous fluid between two concentric rotating cylinders of radii $R_1, R_2(R_1,$. radius of the inner cylinder; R_2, radius of the outer cylinder) with angular velocities Ω_1 andΩ_2 and temperatures T_1 and T_2 respectively. Assuming stationary marginal state, the following equations can be shown to govern the stability of the flow in a narrow-gap annular-space (Soundalgekar *et al.*[4])

$$(D^2 - a^2)\, u = -a^2\, Ta[1 + \alpha x + N(1 + \alpha x)^2]\, v \qquad \dots (1)$$

$$(D^2 - a^2)\, v = u \qquad \dots (2)$$

and the boundry conditions are

$$u = 0,\ Du = 0,\ v = 0 \text{ at } x = 0 \text{ and } 1. \qquad \dots (3)$$

The non-dimensional quantities are defined as follows :

$$d = R_2 - R_1,\ x = (r - R_1)/d,\ a = \lambda d,\ \mu = \Omega_2/\Omega_1,\ \alpha = \mu - 1,\ N = Ra/Ta$$

$$u = \Omega_1 R_1 u',\ \beta = (T_2 - T_1)/d,\ v = 2A\Omega_1 d^2 R_1 v'/\nu,\ \theta = \Omega_1 R_1 d(T_2 - T_1)\, \theta'/k$$

$$Ra = \Omega_1^2 d^3 R_1 \alpha' (T_2 - T_1)/\nu K,\ Ta = -4Ad^4 \Omega_1/\nu^2,\ A = (R_2^2 \Omega_2 - R_1^2 \Omega_1)/(R_2^2 - R_1^2), \qquad \dots (4)$$

Here Ra and Ta are respectively, the Rayleigh number and the Taylor number.

Following Chandrasekhar's analysis we assume a sine series for v as it satisfied the boundary condition (3) on v. So,

$$v = \sum_{m=1}^{\infty} D_m \sin m\pi x. \qquad \dots (5)$$

Substituting eqn. (5) in eqn. (1), we can derive its general solution as follows :

$$u = \sum_{m=1}^{\infty} \frac{D_m}{(m^2\pi^2 + a^2)^2} \left[A_1^{(m)} \cosh ax + B_1^{(m)} \sinh ax + A_2^{(m)} x \cosh ax \right.$$

$$+ B_2^{(m)} x \sinh ax + (1 + \alpha x) \sin m\pi x + \frac{4\alpha n\pi}{m^2\pi^2 + a^2} \cos m\pi x$$

$$\left. +N\left\{(1 + \alpha x)^2 \sin m\pi x + \frac{8\alpha m\pi(1 + \alpha x)}{m^2\pi^2 + a^2} \cos m\pi x + \frac{4\alpha^2(a^2 - 5m^2\pi^2)}{(m^2\pi^2 + a^2)^2} \sin m\pi x\right\}\right] \quad \ldots (6)$$

where $A_1^{(m)}, A_2^{(m)}, B_1^{(m)}, B_2^{(m)}$ are the constants of integration. In view of the boundary conditions (3), we have

$$A_1^{(m)} = -\frac{4\,\alpha m\pi\,(1 + 2N)}{m^2\pi^2 + a^2} \quad \ldots (7)$$

$$A_1^{(m)} \cosh a + B_1^{(m)} \sinh a + A_2^{(m)} \cosh a + B_2^{(m)} \sinh a$$

$$= \frac{(-1)^{m+1}\; 4m\pi\alpha\,[1 + 2N(1 + \alpha)]}{m^2\pi^2 + a^2} \quad \ldots (8)$$

$$a\,B_1^{(m)} + A_2^{(m)} = -\,m\pi\left[1 + N\left\{1 + \frac{12\,\alpha^2\,(a^2 - m^2\pi^2)}{(m^2\pi^2 + a^2)^2}\right\}\right] \quad \ldots (9)$$

$$A_1^{(m)}\; a \sinh a + B_1^{(m)}\, a \cosh a + A_2^{(m)} (\cosh a + a \sinh a) + B_2^{(m)} (\sinh a + a \cosh$$

$$= (-1)^{m+1}\, m\pi\left[(1 + \alpha) + N\left\{(1 + \alpha)^2 + \frac{12\,\alpha^2\,(a^2 - m^2\pi^2)}{(m^2\pi^2 + a^2)^2}\right\}\right]. \quad \ldots (10)$$

We now solve eqns. (7) - (10) and get

$$A_1^{(m)} = -\frac{4\,\alpha m\pi(1 + 2N)}{m^2\pi^2 + a^2} \quad \ldots (11)$$

$$B_1^{(m)} = \frac{m\pi}{\Delta}\left[\, a\,\alpha_m + \beta_m\,(\sinh a + a \cosh a) - \gamma_m \sinh a \,\right] \quad \ldots (12)$$

$$A_2^{(m)} = -\frac{m\pi}{\Delta}\left[\, \alpha_m \sinh^2 a + \beta_m\, a(\sinh a + a \cosh a) - \gamma_m\, a \sinh a \,\right] \quad \ldots (13)$$

$$B_2^{(m)} = \frac{m\pi}{\Delta}\left[\, \alpha_m\,(\sinh a - a) + \beta_m\, a^2 \sinh a - \gamma_m\,(a \cosh a - \sinh a) \,\right] \ldots (14)$$

where

$$\Delta = \sinh^2 a - a^2$$

$$\alpha_m = 1 + N + \frac{12\,\alpha^2 N(a^2 - m^2\pi^2)}{(m^2\pi^2 + a^2)^2}$$

$$\beta_m = \frac{4\alpha}{m^2\pi^2 + a^2}\,[(-1)^{m+1}\{1 + 2N(1 + \alpha)\} + (1 + 2N)\cosh a]$$

$$\gamma_m = (-1)^{m+1}(1+\alpha)\,[1 + N(1+\alpha)] + \frac{(-1)^{m+1}\,12\alpha^2 N(a^2 - m^2\pi^2)}{(m^2\pi^2 + a^2)^2} + \frac{4\alpha a\,(1 + 2N)\sinh a}{m^2\pi^2 + a^2}.$$

Inserting the values of u and v from eqns. (6) and (5) respectively, in eqn. (2), we get

$$\sum_{n=1}^{\infty} D_n\,(n^2\pi^2 + a^2)\sin n\pi x = a^2\,Ta \sum_{m=1}^{\infty} \frac{D_m}{(m^2\pi^2 + a^2)^2}$$

$$\Big[A_1^{(m)}\cosh ax + B_1^{(m)}\sinh ax + A_2^{(m)}x\cosh ax + B_2^{(m)}\,x\sinh ax$$

$$+ (1 + \alpha x)\sin m\pi x + \frac{4\alpha m\pi}{m^2\pi^2 + a^2}\cos m\pi x + N\{(1 + \alpha x)^2\sin m\pi x$$

$$+ \frac{8\,\alpha m\pi(1+\alpha x)}{m^2\pi^2 + a^2}\cos m\pi x + \frac{4\alpha^2\,(a^2 - 5m^{2}pi^{2})}{(m^2\pi^2 + a^2)^2}\sin m\pi x\}]. \qquad \text{... (15)}$$

We now multiply eqn. (15) by sin $m\pi x$ and integrate over the range $0 \le x \le 1$ which yields a system of linear homogeneous equations for the constants and the requirement that these constatns are to all zero leads to the secular equation

$$\Bigg\| \frac{n\pi}{n^2\pi^2 + a^2}\Big[\Big\{1 + (-1)^{n+1}\cosh a\Big\}A_1^{(m)} + (-1)^{n+1}B_1^{(m)}\sinh a$$

$$+ (-1)^{n+1}\Big\{\cosh a - \frac{2a\sinh a}{n^2\pi^2 + a^2}\Big\}A_2^{(m)} + [(-1)^{n+1}\sinh a$$

$$- \frac{2a}{n^2\pi^2 + a^2}\{1 + (-1)^{n+1}\cosh a\}]$$

$$B_2^{(m)}\Big] + \frac{1}{2}\Big[1 + N\Big\{1 + 4\alpha^2\frac{(a^2 - 5m^2\pi^2)}{(m^2\pi^2 + a^2)^2}\Big\}\Big]\delta_{mn}$$

$$+ \alpha X_{mn} + \alpha^2 N\,Y_{mn} - \frac{(n^2\pi^2 + a^2)^3}{2a^2\,Ta}\,\delta_{mn}\Bigg\| = 0 \qquad \text{... (16)}$$

where

$$X_{mn} = \begin{cases} 0 & \text{if } m+n \text{ is even and } m \neq n \\ \frac{1}{4}(1+2N) & \text{if } m = n; \\ \frac{4mn(1+2N)}{m^2-n^2}\left\{\frac{2}{m^2\pi^2+a^2} - \frac{1}{\pi^2(m^2-n^2)}\right\} & \text{if } m+n \text{ is odd;} \end{cases}$$

$$Y_{mn} = \begin{cases} \frac{4mn}{m^2-n^2}\left\{\frac{2}{m^2\pi^2+a^2} + \frac{1}{\pi^2(m^2-n^2)}\right\} & \text{if } m+n \text{ is even and } m \neq n; \\ \frac{2m^2\pi^2-3}{12m^2\pi^2} - \frac{1}{m^2\pi^2+a^2} & \text{if } m = n; \\ \frac{4mn}{m^2-n^2}\left\{\frac{2}{m^2\pi^2+a^2} + \frac{1}{\pi^2(m^2-n^2)}\right\} & \text{if } m+n \text{ is odd.} \end{cases}$$

We now substitute the values of $A_1^{(m)}, A_2^{(m)}, B_1^{(m)}$ and $B_2^{(m)}$ from (11) - (14) in (16), simplify and we have

$$\left|\left|\frac{4mn\pi^2\alpha(1+2N)}{(n^2\pi^2+a^2)(m^2\pi^2+a^2)}[(-1)^{m+1}-1] + \frac{(-1)^{m+n}8Nmn\pi^2\alpha^2}{(n^2\pi^2+a^2)(m^2\pi^2+a^2)}\right.\right.$$

$$- \frac{2amn\pi^2}{(n^2\pi^2+a^2)^2(\sinh^2 a - a^2)}[\sinh a \cosh a - a\}$$

$$\{1 + (1+\alpha)(-1)^{m+n}\}$$

$$+ (\sinh a - a\cosh a)\{(-1)^{n+1} + (1+\alpha)(-1)^{m+1}\}$$

$$- \frac{4a\alpha\sinh a}{m^2\pi^2+a^2}\{\sinh a + a(-1)^{m+n}\}\{(-1)^{m+n} - 1\}$$

$$+ N[(\sinh a\cosh a - a)\{1 + (1+\alpha)^2(-1)^{m+n}\}$$

$$+ (\sinh a - a\,\text{coshn}\, a)\{(-1))^{n+1} + (1+\alpha)^2(-1)^{m+1}\}$$

$$+ \frac{12\alpha^2(a^2-m^2\pi^2)}{(m^2\pi^2+a^2)^2}[(\sinh a\cosh a - a)\{1 + (-1)^{m+n}\}$$

$$+ (\sinh a - a\cosh a)\{(-1)^{n+1} + (-1)^{m+1}\}]$$

$$+ \frac{8a\alpha\sinh a}{m^2\pi^2+a^2}[\sinh a\{1 - (-1)^{m+n}(1+\alpha)\}$$

$$+ a\{(-1)^{m+1}(1+\alpha) - (-1)^{n+1}\}]]]$$

$$+\frac{1}{2}\left[1+N\left\{1+4\alpha^2\frac{(a^2-5m^2\pi^2)}{(m^2\pi^2+a^2)^2}\right\}\right]\delta_{mn}+\alpha X_{mn}+\alpha^2 N Y_{mn}$$

$$-\frac{(n^2\pi^2+a^2)^3}{2a^2\,Ta}\,\delta_{mn}\;\Big|\Big|=0. \qquad \text{... (17)}$$

The numerical calculation are carried out for various values of N and μ and in each case, the values of Ta with respect to a are obtained by solving algebraic equations. A first approximation to the solution of eqn. (17) is obtained by setting the (1, 1)-element of the matrix equal to zero. Then we get, after little algebraic simplification.

TABLE 1 Values of critical Taylor and wave numbers

μ	N	a_c	Ta_c (3rd approx.)	Ta_c(TSA, 1990)
1	–0.5	3.12	3415.85	3415.7
	0.0	3.12	1707.93	1707.8
	0.5	3.12	1138.63	1138.6
	1.0	3.12	853.96	853.9
	1.5	3.11	683.17	683.1
	2.0	3.11	569.31	569.3
	2.5	3.11	487.99	-
	3.0	3.12	426.98	-
0.5	–0.5	3.11	3671.54	3668.4
	0.0	3.12	2275.32	2275.2
	1.0	3.11	1639.04	1648.6
	1.0	3.11	1278.94	1292.6
	1.5	3.11	1047.96	1063.0
	2.0	3.11	887.43	902.7
	2.5	3.11	769.46	-
	3.0	3.11	679.06	-
0.0	–1.0	3.07	7689.02	7623.3
	–0.5	3.13	4743.85	4700.8
	0.0	3.13	3390.48	3390.0
	0.5	3.11	2686.34	2649.1
	1.0	3.10	2207.69	2173.4
	1.5	3.09	1873.89	1892.3
	2.0	3.09	1627.70	1598.0

$$Ta = [6\pi^2(\pi^2+a^2)^5(\sinh a + a)]/[\,a^2\,[\,3\pi^2(2+\alpha)\{(\pi^2+a^2)^2(\sinh a + a)$$

$$-\ 16a\pi^2\cosh^2(a/2)\}\ +\ N[(\sinh a + a)$$

$$\{(\pi^2+a^2)^2(2\pi^2a^2 - 3\alpha^2 + 6\alpha\pi^2 + 6\pi^2)$$

$$-\ 48\pi^4a^2\}\ -\ 24\pi^4 a\ \{2\ \cosh^2\ (1/2)$$

$$\left(2 + 2\alpha + \alpha^2\frac{24\alpha^2(a^2-\pi^2)}{(\pi^2+a^2)^2}\right) - \frac{8a\alpha^2\sinh a}{\pi^2+a^2}\ \}]]]. \qquad \dots (18)$$

When $N \to 0$, equation (18) reduces to

$$Ta = \frac{2}{2+\alpha}\cdot\frac{(\pi^2+a^2)^3}{a^2\,[1 - 16\pi^2 a\cosh^2(a/2)/\{(\pi^2+a^2)^2(\sinh a + a)\}]} \qquad \dots (19)$$

which is same as derived by Chandrasekhar (1961). The numerical values of Ta_c computed from eqn. (17) are listed in Table I.

We observe from this Table I that when N is positive and two cylinders are corotating, the value of Ta_c is observed to decrease with increasing N indicating destabilization of the flow for the same value of N. When only the inner cylinder is rotating with the outer cylinder stationary, the destibilization effect of the increasing N is not so effective as in case of corotating cylinders. When N is negative i.e., the temperature of the outer cylinder is less than that of the inner cylinder, the flow becomes more stable as compared to that in case of $N = 0$ i.e., in the absence of the radial temperature gradient. This shows that by maintaining the temperature of the outer cylinder at a lower level that of the inner cylinder, the flow can be made to remain stable which is an important advantage of the presence of the radial temperature gradient.

3. Conclusions

By keeping the temperature of the outer rotating or stationary cylinder at a level lower than that of the rotating inner cylinder, the flow can be stabilised more and more.

References

1. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Oxford University Press, 1961.
2. S. Chandrasekhar, *J. Rational Mech. Anal.* **3**, (1954), 181-207.
3. D.L. Harris and W.R. Reid, *Fluid Mech.* **20** (1964), 95-110.
4. V.M. Soundalgekar, H.S. Takhar and T.J. Smith, Warme and Stoffubertagung, **15** (1981), 233-38.
5. H.S. Takhar, T. J. Smith and V. M. Soundalgekar, *J. Math. Anal. Applic* **111** (1985), 349-52.
6. H.S. Takhar, V.M. Soundalgekar and M.A. Ali, *J. Math. Anal. Applic.* **152**, (1990), pp. 156-175.
7. G.I. Taylor, *Phil. Trans. Soc. Lond.* **A223** (1923), 289-343.
8. J. Walowit, S. Tsao and R.C. Diprima, *ASME J. Appl. Mech.* **31** (1964), 585-95.