Indian J. pure appl. Math., 25(5); 463-471, May 1994

ON THE GLOBAL CONTROLLABILITY OF A MINIMUM TIME CONTROL PROBLEM FOR A MINIMUM COST CONTROL PROBLEM IN BANACH SPACE

R. N. MUKHERJEE

Department of Mathematics, Burdwan University, Burdwan (W.B.)

(*Received* 3 *December* 1991; *after revision* 23 *November* 1992; *accepted* 14 *December* 1993)

In this paper the global controllability of a certain class of minimum time control problems for minimum cost control problems is defined. Some necessary and sufficient conditions for global controllability of such problems are obtained. An example is given to show the application of the theory.

1. INTRODUCTION

Porter and William[14,15] considered the minimum effort control problem by function space method. Minamide and Nakamura[12] solved a problem which includes as special cases the minimum effort problems. Burns[1] generalized the idea of Minamide and Nakamura[12].

The minimum time control problems were discussed by different authors[3,4,6,8-10]. Chaudhuri and Mukherjee[3, 4] considered a class of time optimal control problems, where the solution of the problems were obtained from that of the auxiliary problem of minimization of norms for a terminal time given in advance.

In this paper the author has obtained the time optimal control from that of auxiliary problem of minimization of the functional, which in this case is the objective function of the minimum cost control problem of Minamide and Nakamura[12]. In order to show the motivation for obtaining such results, important particular cases (when the functional and the constraint set are identified in terms of the norms) are also given.

An example is cited, in short, for application of such results.

Some necessary and sufficient conditions for global controllability of such problems are discussed.

2. SOME PRELIMINARIES

Here we shall use some notations which agree with Minamide and Nakamura[12].

Let B be a real Banach space and B' be its conjugate. The unit ball and unit sphere of B will be denoted by U_B and ∂U_B respectively. Let $K \subset B$ be a convex set. For every $\varphi \in B'$, the number $\langle K, \varphi \rangle$ be defined by $\langle K, \varphi \rangle = \sup_{x \in K} \langle x, \varphi \rangle$ and suppose φ attains its supremum $\langle K, \varphi \rangle$ on K at the vector $x_0 \in K$. We shall denote by $[\varphi : K]$ the set of all such vectors and call it an extremal of φ with respect to K. In particular $[\varphi : U_B]$ will be denoted by $\bar{\varphi}$, usually called an extremal of φ (see Porter[14], Porter and William[15]). For convenience we shall identify a suitable element $x \in [\varphi : K]$ with the set $[\varphi : K]$ itself. So, $[\varphi : K]$ indicates a member or a set. Also let $B_1 \times B_2$ be a product Banach space equipped with the usual product topology. Let $K \subset B_1 \times B_2$ be a convex set. If $(\varphi_1, \varphi_2) \in (B_1 \times B_2)' = B_1' \times B_2'$, then one can write $([\varphi_1 : K], [\varphi_2 : K]) \triangleq [(\varphi_1, \varphi_2) : K]$.

3. Problem Statement

Let X_t (X for all t), Y and Z be Banach spaces. $T_t : X_t$ into Y (T for all t), $S_t : X_t$ onto Z (S for all t), be two bounded linear transformations. Let $\Omega_t \subset X_t$ be a closed convex body containing the origin in its interior. Also let $J(.,.)$ be continuous convex functional defined on $X_t \times Y$, such that $J(x, y) \geq 0$ for all $(X, Y) \in X_t \times Y$, $J(0, 0) = 0$ and $J(x, y) \to +\infty$ as $||x|| + ||y|| \to \infty$ ($||x|| + ||y||$, denotes, the norm in $X_t \times Y$). Let us define the set $J(\alpha)$ by $J(\alpha) = \{(x, y)/J(x, y) \leq \alpha, (x, y) X_t \times Y\}$ and denote by $J(\alpha)$ the boundary of $J(\alpha)$. Clearly for $\alpha > 0$, $J(\alpha)$ is a closed convex body and $\delta J(\alpha) = \{(x, y)/J(x, y) = \alpha, (x, y) X_t \times Y\}$. We consider the mapping the $\bar{T}_t : (X_t \times Y) \to (Y \times Z)$ defined by $\bar{T}_t(u_t, y) = (T_t u_t + y, S_t u_t)$ where $(u_t, y) \in X_t \times Y$, for some given $y \in Y$. Evidently $\bar{T}_t$ ($\bar{T}$ for all t) is linear and onto and $X_t \times Y, Y \times Z$ are Banach spaces and $J(\alpha) \bigcap (\Omega_t \times Y) \subset X_t \times Y$. Let $(\xi, \eta) \in Y \times Z$ be given with $\eta \in \text{Int}(S_t(\Omega_t)$ for some $t \in (0, \infty)$. The problem is to investigate the possibility of reaching any given point (ξ, η) by applying a pair $(u_t, y) \in J(\alpha) \bigcap (\Omega_t \times Y)$ such that t is the minimum time taken for as given $\alpha > 0$, with $\bar{T}_t(u_t, y) = (\xi, \eta)$. In the problems with usually arise in practice, $X_t \times Y$ is an increasing function of t in the sense that $X_{t_1} \times Y \subseteq X_{t_2} \times Y$ whenever $t_1 \leq t_2$. Also $\bar{T}_{t_1}$ can be regarded as the restriction of $\bar{T}_{t_2}$ defined on $X_{t_2} \times Y$, on $X_{t_1} \times Y$. It is easy to show under the above conditions $J(\alpha_1) \bigcap (\Omega_{t_1} \times Y) \subset J(\alpha_2) \bigcap (\Omega_{t_2} \times Y)$ where $\alpha_1 < \alpha_2$.

4. Some Necessary Definitions and Theorems

To solve the problem let us introduce certain definitions and state certain theorems. The proofs of the theorems will be found in Mukherjee[13].

Definition — The set of all points $(\xi, \eta) \in Y \times Z$ such that $\bar{T}_t(u_t, y) = (\xi, \eta)$, $S_t u_t = \eta$ for some $(u_t, y) \in J(\alpha) \bigcap (\Omega_t \times Y)$, and for some $\alpha > 0$, will be called the

Reachable set with respect to the linear transformation $\bar{T}_t$ and will be denoted by $C_\alpha(t)$.

Theorem 1 — The Reachable set is bounded and a convex body.

Corollary — $C_\alpha(t)$ is closed, when X_t is a reflexive space. See Note 1 in Mukherjee[13] to find out the conditions when $C_\alpha(t)$ also closed, if X_t is not a reflexive space, but a conjugate of some other Banach space. To solve the minimum time optimal control problem we shall first consider the following auxiliary problem.

Auxiliary Problem

Let $(\xi, \eta) \in \delta\ \bar{T}_t\ (J(\alpha)) \bigcap (\Omega_t \times Y)$, with η Int $(S_t(\Omega_t))$ for some given time t. Then determine $u_t \in \Omega_t$, such that $\bar{T}_t(u_t, y) = (\xi, \eta)$ and $J(u_t, \xi - T_t u_t)$ is minimized, where $(u_t, y) \in J(\alpha) \bigcap (\Omega_t \times Y)$. The corresponding u_t will be called an optimal control.

Important Particular Cases[12].

Problem (P_1) : $\min\limits_{||u|| \le \rho} ||\xi - Tu||$ subject to $Su = \eta$ $(0 < \rho < +\infty)$.

Problem (P_2) : $\min\limits_{||u|| \le \rho} \{ ||u||^p + ||\xi - Tu||^p \}$ subject to $Su = \eta$ $(0 < \rho < +\infty,\ 0 < \rho \le +\infty)$.

Definition — A control $u_t \in \Omega_t$ will be called an admissible control if for that u_t, $(u_t, y) \in J(\alpha) \bigcap (\Omega_t \times Y)$ i.e. $J(u_t, y) \le \alpha$.

Theorem 2 — An admissible control which will be optimal in the above sense, must satisfy $J(u_t, \xi - T_t u_t) = \alpha$ i.e. $J(u_t, y) = \alpha$.

Theorem 3 — Let $(\xi, \eta) \in \delta\ \bar{T}_t\ (J(\alpha) \bigcap (\Omega_t \times Y))$ and $(\varphi_1, \varphi_2) \ne (0, 0) \in (Y \times Z)*$ denote the supporting hyperplane at (ξ, η). Then $\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle = \langle J(\alpha) \bigcap (\Omega_t \times Y), (T_t' \varphi_1 + S_t' \varphi_2 ; \varphi_1) \rangle$ where T_t' and S_t' denote the conjugate or adjoint transformations to T_t and S_t respectively.

Theorem 4 — If $\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle = \langle J(\alpha) \bigcap (\Omega_t \times Y), (T_t' \varphi_1 + S_t' \varphi_2 ; \varphi_1) \rangle$ for some $(\xi, \eta) \in C_\alpha(t) = \bar{T}_t\ (J(\alpha) \bigcap (\Omega_t \times Y))$ and $(\varphi_1, \varphi_2) \ne (0, 0) \in (Y \times Z)*$, then $(\xi, \eta) \in \delta\ \bar{T}_t\ (J(\alpha) \bigcap (\Omega_t \times Y))$ and (φ_1, φ_2) defines a supporting hyperplane to $C_\alpha(t)$ at (ξ, η) where $\eta \in$ Int $(S_t(\Omega_t))$ and X_t is either a reflexive space or conjugate some other Banach space.

Corollary — Let $(\xi, \eta) \in \delta C_\alpha(t)$ where t is the given terminal time, and $(\varphi_1, \varphi_2) \in (Y \times Z)*$ define a supporting hyperplane at (ξ, η). Let $u_\varphi \in \Omega_t$ be the optimal control such that $\bar{T}_t(u_\varphi, y) = (\xi, \eta)$, $S_t u_\varphi = \eta$, $\eta \in$ Int $(S_t, (\Omega_t))$.

Then $(u_\varphi, y) \in J(\alpha) \bigcap (\Omega_t \times y)$ maximizes $< (u, y), (T_t' \varphi_1 + S_t' \varphi_2; \varphi_1) >$ and u_φ is of the form $u_\varphi = [(T_t' \varphi_1 + S_t' \varphi_2) : J(\alpha) \bigcap (\Omega_t \times Y)]$.

Theorem 5 — The necessary and sufficient condition for the point

$(\xi, \eta) \in C_\alpha(t)$ to be in $\delta C_\alpha(t)$ at the time $t = t_f$ is that

$$\max_{\varphi} \frac{\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_t \times Y), (T_t' \varphi_1 + S_t' \varphi_2 ; \varphi_1) \rangle} = 1.$$

When t_f is the minimum time and X_t is either a reflexive space or it can be considered as the conjugate of some other Banach space.

Theorem 6 — Let $(\xi, \eta) \in C_\alpha(t_f) \bigcap \delta C_\alpha(t_f)$ where $C_\alpha(t_f)$ is the Reachable region. Then

$$\max_{\psi} \frac{\langle (\xi, \eta), (\psi_1, \psi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_{t_f} \times Y), (T_{t_f}' + S' \psi_1 + S_{t_f}' \psi_2 ; \psi_1) \rangle} \leq 1$$

or, ≥ 1, where $\psi = (\psi_1, \psi_2) \in Y^* \times Z^*$, according as $t \geq$ or $\leq t_f$. Moreover the max is attained at a point $\varphi = (\varphi_\xi, \varphi_\eta) \in Y^* \times Z^*$ where φ defines the supporting hyperplane to $\delta C_\alpha(t_f)$ at the intersection with the ray through (ξ, η).

Theorem 7 — Let $t_1 < t_2$ and $\overline{T_{t_1}} : X_{t_1} \times Y \to Y \times Z$, $\overline{T_{t_2}} : X_{t_2} \times Y \to Y \times Z$ be bounded linear onto transformations. Then $C_\alpha(t_1) \leq C_\alpha(t_2)$ and $\delta C_\alpha(t_1) \delta C_\alpha(t_2) = \phi$ if and only if

$$\langle J(\alpha) \bigcap (\Omega_{t_2} \times Y), (T_{t_2}' \varphi_1 + S_{t_2}' \varphi_2 ; \varphi_1) \rangle > \langle J(\alpha) \bigcap (\Omega_{t_1} \times Y) (T_{t_1}' \varphi_1 + S_{t_1}' \varphi_2, \varphi_1) \rangle$$

where $\varphi = (\varphi_1, \varphi_2) \in (Y \times Z)^* = Y^* \times Z^*$ and φ denotes the null set.

Theorem 8 — Let $(\xi, \eta) \in \delta C_\alpha(t_f) \bigcap C_\alpha(t_f)$ and $t \geq t_f$.

Then $$\max_{\varphi = (\varphi_1, \varphi_2)} \frac{\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_t \times Y), \ (T_t' \varphi_1 + S_t' \varphi_2 ; \varphi_1) \rangle}$$

is a non-increasing function of t, for $t \geq t_f$

Corollary — $$\max_{\varphi = (\varphi_1, \varphi_2)} \frac{\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_t \times Y), \ (T_t' \varphi_1 + S_t' \varphi_2 ; \varphi_1) \rangle}$$

is a non-increasing function of t, for $t \geq 0$.

See Note 2 in (13), which theorems are valid without any additional assumption on X_t or $Y \times Z$, such as reflexivity etc.

Note : It follows from above, that the solution of the auxiliary Problem leads to the solution of the time optimal control problem. The above procedure of solving the time optimal control problem will evidently be valid so long as $C_\alpha(t)$ is an increasing function t in the sense $C_{\alpha_1}(t_1) \subset C_{\alpha_2}(t_2)$ whenever $t_1 < t_2$, $\alpha_1 < \alpha_2$ and $\delta C_{\alpha_1}(t_1) \bigcap \delta C_{\alpha_2}(t_2) = \varphi$ and assume global controllability i.e. given any point $(\xi, \eta) \in (Y \times Z)$ there exists at such that $(\xi, \eta) \in \delta C_\alpha(t)$, for some given $\alpha > 0$.

Example : The simplest problem in the calculus of variations is that of finding, in a class of arcs : $x(t)$ $(t_0 \leq t \leq t_1)$ joining two fixed points $x(t_0) = x_0$ and $x(t_1) = x_1$, one which minimizes an integral of the form $J(\dot{x}, x) = \int_{t_0}^{t_1} f(x(t), x(t), t)\, dt$, $(\dot{x}(t) = dx(t)/dt)$.

For function space formulation of the above problem, see 12

5. Solution of the Problem

Let us now consider the question of Global controllability of the system. For this, let $(\xi, \eta) \notin C_\alpha(t)$ and let $(\xi_1, \eta_1) \in \delta C_\alpha(t)$ be on the ray through (ξ, η) i.e. $(\xi_1, \eta_1) = l(\xi, \eta)$ where $0 < l < 1$ and t^* be the minimum time to reach (ξ_1, η_1)

Hence by Theorem 5, we can write

$$\max_{\varphi = (\varphi_1, \varphi_2)} \frac{\langle (\xi_1, \eta_1), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_{t^*} \times Y) ; (T_t^{'} * \varphi_1^{'} + S_t^{'} * \varphi_2 ; \varphi_1) \rangle} \qquad \text{... (A)}$$

where $(\xi_1, \eta_1) \in \delta C_\alpha(t*) \bigcap \delta C_\alpha(t)$.

Suppose maximum is attained at $\varphi = \varphi' = (\varphi_1', \varphi_2')$.

Then $\langle (\xi_1, \eta_1), (\varphi_1', \varphi_2') \rangle = \langle J(\alpha) \bigcap (\Omega_t^* XY), (T_t^{*'} \varphi_1' + S_t^{*'} \varphi_2'; \varphi_1' \rangle > 0$.

Now $\langle (\xi_1, \eta_1), (\varphi_1, \varphi_2) \rangle$ is a continuous function of φ and since $\langle (\xi_1, \eta_1), (\varphi_1' \cdot \varphi_2') \rangle > 0$, there exists neighbourhood of $\varphi' = (\varphi_1', \varphi_2')$, such that $\langle (\xi_1, \eta_1), (\varphi_1, \varphi_2) \rangle > 0$ for all φ in the neighbourhood of φ'. Put $\langle (\xi_1, \eta_1), (\varphi_1, \varphi_2) \rangle = k_\varphi > 0$ in this neighbourhood. Thus $\langle (\xi_1, \eta_1), \frac{1}{k_\varphi} (\varphi_1, \varphi_2) \rangle = 1$. Put $\psi = (\psi_1, \psi_2) = \frac{1}{k_\varphi} (\varphi_1, \varphi_2)$ in (A). Then from (A), we have

$$\frac{1}{\min\limits_{\psi = (\psi_1, \psi_2)} \langle J(\alpha) \bigcap (\Omega_{t*} \times Y) ; (T_{t*}^{'} \psi_1 + S_{t*}^{'} \psi_2 ; \psi_1) \rangle} = 1$$

under the constraint $\langle (\xi_1, \eta_1), (\psi_1, \psi_2) \rangle = 1$

Then the minimum root of the equation

$$\min_{\psi = (\psi_1, \psi_2)} \langle J(\alpha) \bigcap (\Omega_{t*} \times Y), (T_{t*}^{'} \psi_1 + S_{t*}^{'} \psi_2, \psi_1) \rangle = 1. \qquad \text{... (B)}$$

Where $\langle (\xi_1, \eta_1), (\psi_1, \psi_2) \rangle = 1$, will give the minimum time to reach at (ξ_1, η_1).

Now $(\xi_1, \eta_1) \in \delta C_\alpha(t*)$, where t^* is taken as the minimum root of (B). Evidently t^* is the minimum time to reach (ξ_1, η_1). Let $u_{t*} \in \Omega_{t*} \subset X_{t*}$ be the optimal control to reach $(\xi_1, \eta_1) \in \delta C_\alpha(t*)$ i.e. $(\xi_1, \eta_1) = \overline{T}_{t*} (u_{t*}, y)$.

Hence $l(\xi,\eta) = \overline{T}_{t^*}(u_{t*}, y)$. So, in order to reach (ξ,η) in time t^* we shall have to apply the control $u_{t*}/l = v_{t*}$, where $J(v_{t*}, y') = \alpha/l > \alpha$ and $y' = y/l$. Obviously $v_{t*} \notin \Omega_{t*}$. Now let t^{**} be the minimum time to reach (ξ,η) by applying an admissible control, if such a control exist. Then $t^{**} > t^*$. For if possible let $t^{**} < t^*$. Evidently $t^* \neq t^{**}$, as in that case $(\xi,\eta) \in \delta C_\alpha(t*)$, which is not true.

So, let $t^{**} < t^*$. Then by Theorem 6, we can write

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle(\xi_1,\eta_1),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t**}\times Y),\ (T'_{t**}\varphi_1 + S'_{t**}\varphi_2;\varphi_1)\rangle} > 1$$

where $(\xi_1,\eta_1) \in \delta C_\alpha(t^*)$.

But $$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle(\xi_1,\eta_1),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t**}\times Y);\ (T'_{t**}\varphi_1 + S'_{t**}\varphi_2,\varphi_1)\rangle}$$

$$= \frac{\langle(\xi_1,\eta_1),(\varphi_1',\varphi_2')\rangle}{\langle J(\alpha)\bigcap(\Omega_{t**}\times Y);\ (T'_{t**}\varphi_1' + S'_{t**}\varphi_2',\varphi_1')\rangle}$$

$$= \frac{\langle l(\xi,\eta),(\varphi_1',\varphi_2')\rangle}{\langle J(\alpha)\bigcap(\Omega_{t**}\times Y);\ (T'_{t**}\varphi_1' + S'_{t**}\varphi_2',\varphi_1')\rangle}$$

$$= \frac{l\langle(\xi,\eta),(\varphi_1',\varphi_2')\rangle}{\langle J(\alpha)\bigcap(\Omega_{t**}\times Y);\ (T'_{t**}\varphi_1' + S'_{t**}\varphi_2',\varphi_1')\rangle}$$

$= l < 1$ (by Theorem 5), which contradicts (C). Consequently, $t^{**} \leq t^*$ and hence our assertion $t^{**} > t^*$ is correct. Thus we have the following theorem.

Theorem 9 — Let $\overline{T}_t(J(\alpha))\bigcap(\Omega_t\times Y) = C_\alpha(t)$ for any given time t, and let $(\xi,\eta) \notin C_\alpha(t)$. Let $(\xi_1,\eta_1) \in \delta C_\alpha(t)$ be the point on the ray through (ξ,η), and t^* be the minimum time to reach (ξ_1,η_1). If there exists an optimal control $u_t \in \Omega_t$ to reach (ξ,η) in minimum time t^{**}, then $t^{**} > t^*$.

Again from Theorem 6, for $t = t^*$

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle(\xi_1,\eta_1),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t*}\times Y);\ (T'_{t*}\varphi_1 + S'_{t*}\varphi_2,\varphi_1)\rangle} = 1$$

i.e. $$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle l(\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t*}\times Y);\ (T'_{t*}\varphi_1 + S'_{t*}\varphi_2;\varphi_1)\rangle} = 1$$

i.e. $$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle(\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t*}\times Y);\ (T'_{t*}\varphi_1 + S'_{t*}\varphi_2;\varphi_1)\rangle} = \frac{1}{l} > 1$$

where $0 < l < 1$.

Evidently $\max_{\varphi=(\varphi_1,\varphi_2)} \dfrac{\langle (\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_t\times Y);\ \ (T_t'\varphi_1+S_t'\varphi_2\,;\varphi_1)\rangle}$

is also a non-increasing function of t.

Now, if there exists a time $t = t'$, such that

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle (\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t'}\times Y);\ \ (T_{t'}'\varphi_1+S_{t'}'\varphi_2\,;\varphi_1)\rangle} < 1 \text{ and also if}$$

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle (\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t'}\times Y);\ \ (T_{t'}'\varphi_1+S_{t'}'\varphi_2\,;\varphi_1)\rangle} \quad \ldots \text{(C)}$$

is a continuous function of t, then by intermediate value property we can assert that there exists a time $t = t^{**}$ such that

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle (\xi,\eta),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y),\ \ (T_{t^{**}}'\varphi_1+S_{t^{**}}'\varphi_2\,;\varphi_1)\rangle} = 1. \quad \ldots \text{(D)}$$

Now, $(\xi,\eta)\in C_\alpha(t^{**})$. For if $(\xi,\eta)\notin C_\alpha(t^{**})$ let $(\xi',\eta')\in\delta C_\alpha(t^{**})$ be the point on the ray through (ξ,η) so that $(\xi',\eta') = l(\xi,\eta)$ for some $l < 1$.

Hence from (D), we obtain

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle (\xi',\eta'),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y),\ \ (T_{t^{**}}'\varphi_1+S_{t^{**}}'\varphi_2\,;\varphi_1)\rangle} = l < 1$$

which contradicts

$$\max_{\varphi=(\varphi_1,\varphi_2)} \frac{\langle (\xi',\eta'),(\varphi_1,\varphi_2)\rangle}{\langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y),\ \ (T_{t^{**}}'\varphi_1+S_{t^{**}}'\varphi_2\,;\varphi_1)\rangle} = 1 \text{ (by Theorem 5).}$$

Now, from Theorem 4, $(\xi,\eta)\in\delta C_\alpha(t^{**})$ and the maximum in (D) will be attained at $\psi=(\psi_1,\psi_2)$ which defines the supporting hyperplane to $C_\alpha(t^{**})$ at (ξ,η). Therefore we have

$$\langle (\xi,\eta),(\psi_1,\psi_2)\rangle = \langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y);\ (T_{t^{**}}'\psi_1+S_{t^{**}}'\psi_2;\psi_1)\rangle.$$

Also, since $(\xi,\eta)\in\delta C_\alpha(t^{**})$, there exists a $u_\xi\in\Omega_{t^{**}}$ such that $(\xi,\eta)=\overline{T}_{t^{**}}(u_\xi,y)$.

Hence $\langle \overline{T}_{t^{**}}(u_\xi,y);(\psi_1,\psi_2)\rangle = \langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y);\ (T_{t^{**}}'\psi_1+S_{t^{**}}'\psi_2;\psi_1)\rangle$

or $\langle (u_\xi,y),(T_{t^{**}}'\psi_1+S_{t^{**}}'\psi_2;\psi_1)\rangle = \langle J(\alpha)\bigcap(\Omega_{t^{**}}\times Y);\ (T_{t^{**}}'\psi_1+S_{t^{**}}'\psi_2;\psi_1)\rangle$.

Consequently, by Hahn-Banach Theorem u_ξ can be chosen to be

$$u_\xi = [\,T_{t^{**}}'\psi_1+S_{t^{**}}'\psi_2 : J(\alpha)\bigcap(\Omega_{t^{**}}\times Y)] \text{ with } J(u_\xi,y)=\alpha.$$

It can be similarly proved for $(\xi, \eta) \in \text{Int}\, C_\alpha(t)$.

Thus we obtain the following results.

Theorem 10 — The sufficient conditions for the existence of minimum time control for (ξ, η) as in Theorem 9, are

(a) there exists a time t_1, such that

$$\max_{\varphi = (\varphi_1, \varphi_2)} \frac{\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_{t_1} \times Y),\ (T_{t_1}{}' \varphi_1 + S_{t_1}{}' \varphi_2 ; \varphi_1) \rangle} < 1$$

and (b)

$$\max_{\varphi = (\varphi_1, \varphi_2)} \frac{\langle (\xi, \eta), (\varphi_1, \varphi_2) \rangle}{\langle J(\alpha) \bigcap (\Omega_t \times Y),\ (T_{t_1}{}' \varphi_1 + S_{t_1}{}' \varphi_2 ; \varphi_1) \rangle}$$

is a continuous function of t.

Theorem 11 — The necessary condition for the existence of an admissible optimal control is that

$$\min_{\psi = (\psi_1, \psi_2)} \langle J(\alpha) \bigcap (\Omega_t \times Y), (T_t{}' \psi_1 + S_t{}' \psi_2 ; \psi_1) \rangle = 1$$

under the constraint $\langle (\xi, \eta), (\psi_1, \psi_2) \rangle = 1$ will have at least one real positive root.

6. CONCLUSION

Theorems 10 and 11 state some necessary and sufficient conditions to obtain the minimum time to reach any given (ξ, η) and the corresponding optimal control. It is a routine affairs, by applying the above theory, to obtain the time optimal control and the minimum time for the particular cases as mentioned for the auxiliary problem and the example.

ACKNOWLEDGEMENT

The author is thankful to Professor A. K. Chaudhuri of Indian Institute of Management, Calcutta, India, for his guidence to formulate and to solve the problem in this paper.

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