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ON EXTENSION OF CERTAIN MAP FROM A TOPOLOGICAL SPACE TO A BITOPOLOGICAL SPACE

S. K. SEN AND M. N. MUKHERJEE

Department of Pure Mathematics, University of Calcutta
35, Ballygunge Circular Road, Calcutta 700 019

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In the present paper, we derive a few necessary and sufficient conditions, under which a certain type of functions from a topological space X to a bitopological space Y can be extended to any extension space X^* of X. The deliberations ultimately lead us to achieve, as a particular case, the famous Taimanov's[12] theorem on extension of continuous maps.

One of the fundamental theorems of the theory of extensions of continuous maps on a topological space X over an extension X^* of X, was given by Taimanov[12]. The theorem asserts that a continuous map $f : X \to Y$ with Y a compact, Hausdorff space, can be extended to a continuous map onto any extension X^* of X iff for any two disjoint closed sets A and B in Y, the closures of $f^{-1}(A)$ and $f^{-1}(B)$ in X^* are disjoint. Different generalizations of this theorem involving extensions of certain continuous-like maps, have resulted so far (e. g. see Mcdowell[5], Rudolf[8], Velicko[13]). It is the purpose of this paper to derive certain characterizing conditions under which a generalized type of weakly continuous function from a topological space X to a bitopological space Y can be extended over an arbitrary extension space of X. In the process we derive Taimanov's theorem as a special case.

Henceforth in what follows, by spaces X and Y we shall mean a topological space (X, T) and a bitopological space[3] (Y, P_1, P_2), the latter space Y being endowed with two arbitrary topologies P_1 and P_2; also whenever i and j both occur in any context, we shall assume that $i, j \in \{1, 2\}$ and $i \neq j$. For a subset A of (X, T) (resp. (Y, P_1, P_2)), T-cl$A$ (resp. P_i-cl$A$) will mean the closure of A in (X, T) (resp. (Y, P_i), for $i = 1, 2$). For any $A \subset Y$, the ij-θ-closure of A, denoted by ij-θ-cl$A$, consists of those points y of Y such that for every P_i-open nbd (neighbourhood) V of y, $(P_j\text{-cl}\, V) \bigcap A \neq \phi$ (Kariofillis[2]). $A(\subset Y)$ is called ij-θ-closed iff $A = ij$-θ-cl$A$. The complements of ij-θ-closed sets in Y (called ij-θ-open sets) form a topology P_i^θ (say) on Y(Banerjee[1]). Y is called pairwise Urysohn[11] iff for any two distinct x, y of Y, there exists a P_i-open nbd U of x and a P_j-open nbd V of y such that P_j-cl$U$

$\bigcap P_i$-cl$V = \phi$. Let (X^*, T^*) be an extension of the space (X, T), then the system of all T-open (T^*-open) nbds of a point x of (X, T) (resp. (X^*, T^*)) will be denoted by N_x (resp. N_x^*). The concept of quasi-H-closedness (QHC) was first generalized to a bitopological space by Mukherjee[6]. Such a space was further studied by Kariofillis[2] and Mukherjee et al.[7]. We begin by recalling a few definitions and results that will be used very often in the sequel.

Definition 1[6] — A space (Y, P_1, P_2) is said to be ij-QHC iff every P_j-open filterbase has a P_i-adherent point; Y is called pairwise QHC iff it is 12-QHC as well as 21-QHC.

Theorem 2[7] — In a pairwise Urysohn, pairwise QHC space (Y, P_1, P_2), for each $y \in Y$ and each P_i-open nbd U of y, there are P_j-open sets A_y and B_y such that $y \in A_y \subset P_i$–cl $A_y \subset B_y \subset P_i$-cl$U$.

Theorem 3[7,10] — In a pairwise Urysohn, pairwise QHC space (Y, P_1, P_2) the following are true :

(a) $P_1^\theta = P_2^\theta$.

(b) ij-θ-cl$B$ = ij-θ-cl (ij-θ-cl$B$), for any $B \subset Y$.

Definition 4 — A point y in a space (Y, P_1, P_2) is called an ij-θ-adherent point of a filterbase $\mathcal{F}$ on Y iff $y \in ij$–θ–cl$F$, for each $F \in \mathcal{F}$. The set of all ij-θ-adherent points of $\mathcal{F}$ is called the ij-θ-adherence of $\mathcal{F}$ and is denoted by ij-θ-ad $\mathcal{F}$. The filterbase $\mathcal{F}$ is said to ij-θ-converge to a point $y \in Y$, written as $\mathcal{F} \overset{ij-\theta}{\rightarrow} y$, iff P_j-closure of every P_i-open nbd of y contains some member of $\mathcal{F}$.

Theorem 5[2] — A space (Y, P_1, P_2) is ij-QHC iff every filterbase on Y has a non-void ij-θ-adherence.

Lemma 6 — A filterbase $\mathcal{F}$ on a pairwise QHC, pairwise Urysohn space Y ij-θ-converges iff it has a unique ij-θ-adherent point.

PROOF : For a filterbase $\mathcal{F}$ on Y ij-θ-converging to some $x \in Y$ and any y of Y with $y \neq x$, there exist a P_i-open nbd U of x and a P_j-open nbd V of y such that P_j-cl$U \bigcap$ P_i-cl$V$ = ϕ. By Theorem 2 we have $y \in W \subset P_j$-cl$W \subset P_i$-cl$V$, for some P_i-open nbd W of y. Now, for some $F \in \mathcal{F}, F \subset P_j$ – cl$U$ so that $F \bigcap P_j$ - cl$W$ = ϕ and hence y is not an ij-θ-adherent point of $\mathcal{F}$.

Conversely, let x be the unique ij-θ-adherent point of a filter-base $\mathcal{F}$ on Y. Let U be a P_i-open nbd of x such that P_j-cl$U$ contains no member of $\mathcal{F}$. Then x is not an ij-θ-adherent point of the filterbase $\mathcal{B}$ = $\{F \bigcap (Y - P_j\text{-cl}U) : F \in \mathcal{F}\}$ so that by Theorem 5, $\mathcal{F}$ has an ij-θ-adherent point other than x, a contradiction.

Definition 7 — Let $f: (X, T) \rightarrow (Y, P_1, P_2)$ be a mapping. f is called ij-weakly continuous iff for each $x \in X$ and each P_i-open nbd V of $f(x)$, there exists a T-open nbd U of x such that $f(U) \subset P_j$-cl$V$.

Remark 8 : If $P_1 = P_2$ (= P, say), then the above definition of ij-weak continuity of a map $f : (X, T) \to (Y, P, P)$ coincides with the usual definition of weakly continuous function of Levine[4].

Theorem 9[9] — A function $f : (X, T) \to (Y, P_1, P_2)$ is ij-weakly continuous iff $f(T\text{-cl}A) \subset ij\text{-}\theta\text{-cl}f(A)$, for each $A \subset X$.

Corollary 10 — If $f : X \to Y$ is ij-weakly continuous, then for any ij-θ-closed set U in Y, $f^{-1}(U)$ is closed in X.

Lemma 11 — A map $f : X \to Y$ is ij-weakly continuous iff for any filterbase $\mathcal{F}$ on X, $f(\mathcal{F}) \overset{ij-\theta}{\to} f(x)$ whenever $\mathcal{F} \to x$ in X.

PROOF : Straightforward and omitted.

Lemma 12[2] — For a space (Y, P_1, P_2), if $U \in P_j$, then $ij\text{-}\theta\text{-cl}U = P_i\text{-cl}U$.

We are now in a position to prove our main theorem of this paper as follows.

Theorem 13 — Let f be an ij-weakly continuous function from a topological space (X, T) to a pairwise QHC, pairwise Urysohn space (Y, P_1, P_2), and let (X^*, T^*) be any extension of X. Then f can be extended to an ij-weakly continuous function f^* on (X^*, T^*) iff for any two disjoint ij-θ-closed subsets B_1 and B_2 of Y, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ have non-interesecting closures in X^*.

PROOF : *Necessity* : Let B_1, B_2 be ij-θ-closed in Y such that $B_1 \cap B_2 = \phi$. If $x \in T^*\text{-cl}f^{-1}(B_1) \cap T^*\text{-cl}f^{-1}(B_2)$, then by Theorem 9, $f^*(x) \in f^*(T^*\text{-cl}\, f^{-1}(B_1)) \subset ij\text{-}\theta\text{-cl}\, f^* f^{-1}(B_1) \subset ij\text{-}\theta\text{-cl}B_1 = B_1$. Similarly $f^*(x) \in B_2$, and hence $B_1 \cap B_2 \neq \phi$, a contradiction. This proves that $T^*\text{-cl}\, f^{-1}(B_1) \cap T^*\text{-cl}f^{-1}(B_2) = \phi$.

Sufficiency : Let $x \in X^*$, and $\mathcal{F}_x = \{U \cap X : U \in N_x^*\}$. Since X is dense in X^*, $f(\mathcal{F}_x)$ is a filterbase on Y. Since Y is ij-QHC, $ij\text{-}\theta\text{-ad}\, f(\mathcal{F}_x) \neq \phi$. We will prove that $ij\text{-}\theta\text{-ad}\, f(\mathcal{F}_x)$ consists of a single point y (say) of Y. Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$, and $y_1, y_2 \in ij\text{-}\theta\text{-ad}\, f(\mathcal{F}_x)$. By pairwise Urysohn property of Y, we choose a P_i-open nbd V_1 of y_1 and a P_j-open nbd V_2 of y_2 such that $P_j\text{-cl}V_1 \cap P_i\text{-cl}V_2 = \phi$. Now by Theorem 2, there exists a P_i-open nbd U_2 of y_2 such that $P_j\text{-cl}U_2 \subset P_i\text{-cl}V_2$, i.e. $P_j\text{-cl}V_1 \cap P_j\text{-cl}U_2 = \phi$ so that $ji\text{-}\theta\text{-cl}V_1 \cap ji\text{-}\theta\text{-cl}U_2 = \phi$ (by Lemma 12). By virtue of Theorem 3, $ji\text{-}\theta\text{-cl}V_1$ and $ji\text{-}\theta\text{-cl}U_2$ are ji-θ-closed as well as ij-θ-closed, and hence we have by hypothesis, $T^*\text{-cl}\, f^{-1}(P_j\text{-cl}V_1) \cap T^*\text{-cl}\, f^{-1}(P_j\text{-cl}U_2) = \phi$. But as $f(F)$ intersects $P_j\text{-cl}V_1$ as well as $P_j\text{-cl}U_2$ for every $F \in \mathcal{F}_x$, any T^*-open nbd of x intersects $f^{-1}(P_j\text{-cl}V_1)$ and $f^{-1}(P_j\text{-cl}U_2)$, i.e., $x \in T^*\text{-cl}\, f^{-1}(P_j\text{-cl}V_1) \cap T^*\text{-cl}\, f^{-1}(P_j\text{-cl}U_2)$. The contradiction shows that $ij\text{-}\theta\text{-ad}\, f(\mathcal{F}_x)$ consists of a single point y. By Lemma 6, $f(\mathcal{F}_x)$ ij-θ-converges to that point y. Thus for each $x \in X$, define $f^*(x) = y$ (as obtained above).

It is clear that for $x \in X$, $f(x) = f^*(x)$. In fact, let $x \in X$. By definition of f^*, $f(\mathcal{F}_x)$ ij-θ-converges to the unique point $f^*(x)$.

But since $f : X \to Y$ is ij-weakly continuous and $x \in X$, we have by Lemma 11 that $f(\mathcal{F}_x) \xrightarrow{ij-\theta} f(x)$, because $\mathcal{F}_x \to x$ in (X, T). Thus $f*(x) = f(x)$, for all $x \in X$.

Let $x \in X^*$ and let us take a P_i-open nbd U of $f^*(x)\,(= y$, say). Since $f(\mathcal{F}_x)$ ij-θ-converges to y, there exists a T^*-open nbd W of x such that $f(V) \subset P_j$-cl$U$, where $V = W \cap X$. Now let $z \in W$, we have by definition of f^*, $f^*(z) = \bigcap \{ij$-θ-cl $f(A \cap X) : A \in N_z^*\} \subset ij$-$\theta$-cl $f(V)$ (since $W \in N_z^*$). Now we have $f(V) \subset P_j$-cl$U$ = ji-θ-cl$U$ (by Lemma 12) = ij-θ-cl (ji-θ-cl$U$) (by Theorem 3). Hence by using Theorem 3 once again we have $f^*(z) \in ij-\theta$-cl (ij-θ-cl(ji-θ-cl$U$)) = ij-θ-cl(ji-θ-cl$U$) = P_j-cl$U$. Thus $f^*(W) \subset P_j$-cl$U$. Hence f^* is an ij-weakly continuous function on X^* to Y.

Remark 14 : Putting $P_1 = P_2$ (= P, say) in the above theorem we can easily deduce Taimanov's theorem. In fact, whenever (Y, P) is compact and Hausdorff, it is Urysohn, QHC and regular. Also it is known that in a regular space a set is closed iff it is θ-closed and that a weakly continuous function from any space to a regular space becomes continuous.

Theorem 15 — An ij-weakly continuous function f from a space (X, T) into a pairwise QHC pairwise Urysohn space (Y, P_1, P_2) can be extended to an ij-weakly continuous function to an extension (X^*, T^*) of X iff for each $x \in X^*$, the image under f of $\mathcal{F}_x = \{U \cap X : U \in N_x^*\}$ has a unique ij-θ-adherent point.

PROOF : Let f have an ij-weakly continuous extension f^* from X^* to Y. From the property of ij-weakly continuous function f^* at $x \in X^*$, it follows by Lemma 11 that $f^*(N_x^*) \xrightarrow{ij-\theta} f^*(x)$ and therefore $f^*(x)$ is the unique ij-θ-adherent point $f^*(N_x^*)$ (by Lemma 6). Hence $f(\mathcal{F}_x)$ has a unique ij-θ-adherent point.

Conversely, for any $x \in X^*$, let $\mathcal{F}_x = \{U \cap X : U \in N_x^*\}$, then $f(\mathcal{F}_x) \xrightarrow{ij-\theta} y$ (say) $\in Y$ (by Lemma 6). Assuming $f^*(x) = y$ and proceeding as in Theorem 13, we get the required ij-weakly continuous extensions.

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