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PROBABILIST, POSSIBILIST AND BELIEF OBJECTS FOR PATTERN RECOGNITION BY DATA ANALYSIS

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The main aim of the symbolic approach in data analysis is to extend problems, methods and algorithms used on classical data to more complex data called "symbolic objects" which are well adapted to representing knowledge and which "unify" unlike usual observations which characterize "individual things". We introduce several kinds of symbolic objects : boolean, possibilist, probabilist and belief. We briefly present some of their qualities and properties; three theorems show how Probability, Possibility and Evidence theories may be extended on these objects. Finally four kinds of data analysis problems including the symbolic extension are presented.

INTRODUCTION

If we wish to describe the fruits produced by a village, by the fact that "The weight is between 300 and 400 grammes and the color is white or red and if the color is white then the weight is lower than 350 grammes", it is not possible to put this kind of information in a classical data table where rows represent villages and columns descriptors of the fruits. This is because there will not be a single value in each cell of the tables (for instance for the weight) and also because it will not be easy to represent rules (if ...., then ....) in this table. It is much easier to represent this kind of information by a logical expression such as :

$$a_i = [\text{weight} = [300, 400]] \wedge [\text{color} = \{\text{red, white}\}] \wedge [[\text{color} = \text{white}] \rightarrow [\text{weight} \leq 350]]$$

where a_i, which represents the ith village, is a mapping defined on the set of fruits such that for a given fruit ω, $a_i(\omega)$ = true iff the weight of ω belongs to the interval [300, 400], its color is red or white and if it is white then its weight is less than 350 gr. Following the terminology of this paper a_i is a kind of symbolic object. If we have a set of 1000 villages represented by a set of 1000 symbolic objects $a_1, \ldots\ldots, a_{1000}$, an important problem is to know how to apply data analysis or statistical methods to it. For instance, what is a histogram, a classification, a discrimination or a probability law for such a set of objects? The aim of symbolic data analysis (Diday[3,4]) is to provide tools for answering this problem.

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In some fields a boolean representation of the knowledge ($a_i(\omega)$ = true or false) is sufficient to get the main information, but in many cases we need to include uncertainty to represent the real world with more accuracy. For instance, if we say that in the ith village "the color of the fruits is often red and seldom white" we may represent this information by a_i = [color = often red, seldom white]. More generally, in the case of boolean objects or objects where frequency appears, we may write a_i = [color = q_i] where q_i is a characteristic function in the boolean case, and a probability measure in the second case. More precisely, in the boolean case, if a_i = [color = red, white] we have q_i (red) = q_i (white) = 1 and q_i = 0, for the other colors; in the probabilist case, if a_i = [color = 0.9 red, 0.1 white] we have q_i (red) = 0.9, q_i (white) = 0.1.

If an expert says that the fruits are red we may represent this information by a symbolic object a_i = [color = q_i] where q_i is a "possibilist" function in the sense of Dubois and Prade (1986); we will have for instance q_i (white) = 0, q_i (pink) = 0.5 and q_i (red) = 1. If an expert who has to study a representative sample of fruits from the ith village, says that 60 per cent are red, 30 per cent are white and the color is unknown for 10 per cent which were too rotten, we may represent this information by a_i = [color = q_i] where q_i is a belief function such that q_i (red) = 0.6, q_i (O) = 1, where O is the set of possible colors. Depending on the kind of the mapping q_i used, a_i has been called a boolean, probabilist, possibilist or belief object. In all the cases a_i is a mapping from Ω in [0, 1]. Now, the problem is to know how to compute $a_i(\omega)$; if there is doubt about the color of a given fruit ω, for instance, if the expert says that "the color of ω, is red or pink" then, ω may be described by a characteristic function r and represented by a symbolic object ω^s = [color = r] such that r (red) = r (pink) = 1 and r = 0 for the other colors. Depending on the kind of knowledge that the user wishes to represent, r may be a probability possibility or belief function. Having a_i = [color = q_i] and ω^s = [color = r] to compute a_i (w) we introduce a comparison function f such that a_i (w) = g (q_i, r) measures the fit between a_i and r. What is the meaning of a_i (ω) ? May we say that a_i (ω) measures a kind of probability, possibility o. belief that w belongs to the class of fruits described by a_i when q_i and r are respectively characteristic, probability, possibility or belief function ? To answer this question we have extended a_i to a "dual" mapping a_i^* (such that $a_i(\omega) = a_i^*(\omega^s)$) defined on the set of symbolic objects of the a_i kind denoted a_x and an extension of the union, intersection and complementary operators of classical sets denoted $OP_x = \{\bigcup_X, \bigcap_X, c_X\}$ where x depends upon the kind of knowledge used; then, we have shown that when x represents probability, then a_i^* satisfies the axioms of probability measures by using OP_{pr} (x = probability) and in the case of possibilist objects that a_i^* satisfies the axioms of possibility functions by using some given operators denoted OP_{pos} (see Diday (1991) for more details).

In probability theory, every little is said about events which are generally identified as parts of the set of samples Ω. In computer science, object oriented languages consider more general events called objects or "frames" defined by intention. In data analysis (multidimensional scaling, clustering, explanatory data analysis etc.) more importance is given to the elementary objects which belong to the sample Ω than in classical statistics where attention is focused on the probability laws of Ω; however, objects of data analysis are generally identified to point of $\mathcal{R}^p$ and hence are unable to treat complex objects coming for instance from large data bases, and knowledge bases. Our aim is to define complete objects called "symbolic objects" inspired by those of oriented object languages in such a way that data analysis becomes generalized in knowledge analysis. Objects will be defined by intention by the properties of their extension. More precisely we distinguish objects which "unify" rather than elementary observed objects which characterize "individual things" (their extension) : for instance "the customers of my shop" instead of " a customer of my shop", "a species of mushroom" instead "the mushroom that I have in my hand".

By extending data analysis methods to symbolic objects this paper makes a bridge between several domains : "data analysis and statistics" (where limited interest has, as yet been shown in treating this kind of object), "statistical data bases" (where symbolic objects may be considered as "metadata" which means data on the data) "management of uncertainty in knowledge-based systems" (where the emphasis is now more on knowledge representation and reason in then on data analysis) "learning machine" (where this kind of objects as input and classical methods of data analysis has been neglected) and more generally in AI (where the results here obtained, in theorem 1, 2, 3, concern metaknowledge or knowledge on knowledge).

We have not used the notion of "predicates" from classical logic firstly, because by using only functions things seem more understandable, especially to statisticians; secondly, because they cannot be used simply in the case of probabilist, possibilist and belief objects where uncertainty is present.

1. Boolean Symbolic Objects

We consider Ω a set of individual things called "elementary objects" and a set of descriptor functions $y_i : \Omega \to O_i$.

A basic kind of symbolic object is "*event*". An event denoted $e_i = [y_i = V_i]$ where $V_i \subseteq O_i$ is a function $\Omega \to \{\text{true, false}\}$ such that $e_i(w)$ true iff $y_i(w) \in V_i$. When $y_i(w)$ is meaningless (the kind of computer used by a company without computer) $V_i = \phi$ and when it has a meaning but this is not known $V_i = O_i$. The extension of e_i in Ω denoted by ext (e_i / Ω) is the set of elements $w \in \Omega$ such that $e_i(w) =$ true.

An assertion is a conjuction of events a = $\hat{i}$ $[y_i + V_i]$; the extension of a denoted ext(a / Ω) is the set of elements of Ω such that $\forall i \; y_i(w) \in V_i$.

A "*horde*" is a symbolic object which appears, for instance, when we need to express relations between parts of a picture that we wish to describe. More generally a horde is a function h from Ωp in {true, false} such that $h(u) = \hat{i}\ [y_i(u_i) = V_i]$ if $u = (u_1, ..., u_p)$. For example : $h = [y_1(u_1)] \wedge [y_2(u_2) = \{3, 5\}] \wedge [y_3(u_1) = [30, 35]] \wedge$ [neighbour (u_1, u_2) = yes].

A synthesis object is a conjunction or a semantic link between hordes denoted in the case of conjunction by $s = \hat{i}\ h_i$ where each horde may be defined on a different set Ω_i by different descriptors. For instance Ω_1 may be individuals, Ω_2 location, Ω_3 kind of job etc. All these objects are detailed in Diday (1991).

2. Modal Objects

Suppose that we wish to use a symbolic object to represent individuals of a set satisfying the following sentence : "It is possible that their weight be between 300 and 500 grammes and their color is often red or seldom white" ; this sentence contains two events e_1 = [color = {red, white}] which lack the modes *possible, often* and *seldom,* a new kind of event denoted f_1 and f_2, is needed if we wish to introduce them f_1 = *possible* [height = 300, 500]] and f_2 = [color = {*often* red, *seldom* hite}]; we can see that f_1 contains an *external* mode *possible* affecting e_1 whereas f_2 contains *internal* modes affecting the values contained in e_2. Hence, it is possible to describe informally the sentence by a modal assertion object denoted $a = f_1 \wedge_X f_2$ where $\wedge_X$ represents a kind of conjunction related to the background knowledge of the domain. The case of modal assertions of the kind $a = \hat{i}\ f_i$ where all the f_i are events with external modes has been studied, for instance, in Diday[3]. This paper is concerned with the case where all the f_i contain only internal modes.

3. Internal Modal Objects

3.1 A formal definition of internal modal objects

Let x be the background knowiedge and

• M^x a set of modes for instance M^x = {often, sometimes, seldom, never} or $M^x = [0, 1]$.

• $Q_i = \{ q_i^j \}_j$ a set of q_i^j from O_i in M^x, for instance O_i = {red, yellow, green},

$M^x = [0, 1]$ and q_i^j (red) = 0.1; q_i^j (yellow) = 0.3; q_i^j (green) = 1, where the meaning of the values 0.1, 0.3, 1 depends on the background knowledge (for instance q_i^j may express a possibility, see §4.1)

• y_i is a descriptor (the *color* for instance); it is a mapping from Ω in Q_i. Notice that in the case of boolean objects y_i was a mapping from Ω in O_i, and not Q_i.

Example — If O_i and M^x are chosen as in the previous example and the color of w is red then $y_i(\omega) = r$ means that $r \in Q_i$ be defined by r (red) = 1, r (yellow) = 0, r (green) = 0.

• $OP_x = \{\bigcup_X, \bigcap_X, c_X\}$ where $\bigcup_X, \bigcap_X$ expresses a kind of union and intersection between subsets of Q_i and $c_x(q_i)$ (sometimes denoted $\bar{q_i}$, the commentary of $q_i \in Q_i$)

Example — If $q_i^j \in Q_i$ and $Q_i^j \subseteq Q_i$

$$q_i^1 \bigcup_X q_i^2 = q_i^1 + q_i^2 - q_i^1 q_i^2$$

$$q_i^1 \bigcap_X q_i^2 = q_i^1 q_i^2 \text{ where } q_i^1 q_i^2 (v)\ q_i^2 (v)\ ;\ c(q_i) = 1 - q_i$$

$$Q_i^1 *_X Q_i^2 = b(Q_i^1) *_X b(Q_i^2) \text{ where} *_X \in \{\bigcup_X, \bigcap_X\} \text{ and}$$

$$b(Q_i^j) = \{\bigcup_X q_i / q_i \in Q_i^j = 1 - c_X(b(Q_i^j)).$$

This choice of OP_X is "archimedian" because it satisfies a family of properties studied by Shweizer and Sklar (1960) and recalled by Dubois et Prade (1988).

• g_X is a "comparison" mapping from $Q_i \times Q_i$ in an ordered space L^X.

Example — $L^X = M^X = [0, 1]$ and $g_X(q_i^1, q_i^2) = < q_i^1, q_i^2 >$ the scalar product

• f_X is an "aggregation" mapping from $P(L^X)$ the power set of L^X in L^X. For instance, $f_X(\{L_1, \ldots\ldots, L_n\}) = \text{Max } L_i$.

Let $Y = \{y_i\}$ be a set of descriptors and $V = \{V_i\}$ a set of subsets of Q_i such that $V_i = \{ q_i^j \} \subseteq Q$. Now we are able to give the formal definition of an internal object (called "im" object).

Definition of an im assertion — Given OP_x, g_x and f_x, an im object is a mapping a_{YV} from Ω in an ordered spaceL^X denoted $a = \hat{i}\ [y_i = \{q_i^j\}_j\ ]$ such that if $\omega \in \Omega$ is described for any i by $y_i(\omega) = \{ r_i^j \}$ then

$$a_{yv}(\omega) = f_x(\{g_x(\bigcup_{x\,j} q_i^j, \bigcup_{x\,j} r_i^j)\}_i).$$

We denote by a_x the set of im objects associated to background knowledge x and φ the mapping from Ω in a_x such that $\varphi(\omega) = \omega^s = \hat{i}_x [y_i = y_i(\omega)]$.

3.2 Extension of im objects

There are at least two ways to define the extension of an im object a. The first consists in considering that each element $\omega \in \Omega$ is more or less in the extension of a according to its weight given by a (ω); in this case the extension of a denoted Ext (a/Ω) will be the set of couples

$\{(\omega, a(\omega)) / \omega \in \Omega\}$. The second requires a given threshold α and then the extension of a will be Ext $(a/\Omega, \alpha) = \{(\omega, a(\omega)/\omega \in \Omega, a(\omega) \geq \alpha\}$.

3.3 Semantic of im objects

In addition to the modes, several other notions may be expressed by an im object a :

a) Certitude : $a(\omega)$ is not true or false as for boolean objects but expresses a degree of certitude.

b) Variation : this appears at two levels in an im object denoted $a = \hat{i}_x [y_i = \{d_i^j\}_i]$; first in each q_i^j, for instance if y_i is the color and q_i^1 (red) = 0.5, q_i^1 (green) = 0.3 it means that a variation exists between the individual objects which belong to the extension of a (for instance a species of mushrooms) where some are red and others are green; second, for given discription y_i between the q_i^j (each q_i^j expresses for instance the variation in a different kind of species).

c) Doubt : if we say that the color of a species of mushroom is red "or" green it is an "or" of variation but if we say that the color of the mushroom which is in my hand is red "or" green, it is an "or" of doubt.

Hence, if we describe $\omega \in \Omega$ by $\varphi(\omega) = \omega^s = \hat{i}\ [y_i = y_i(\omega)]$ where $y_i(\omega) = \{r_i^j\}_j$ we express a doubt in r_i^j and among the r_i^j provided, for instance, by several experts.

3.4 An example of background knowledge expressing "intensity"

Here the background knowledge x is denoted i for intensity. Each individual object $\omega \in \Omega$ is a manufactured object described by two features y_1 which express the degree of "roundness" and "flatness" and y_2 the "heaviness" : O_1 = {flat, rond}, O_2 = {heavy} ; M^i = {*very, quite, a little, very little, nil*}.

Let a and ω^s be defined by :

$$a = [y_1 = \textit{a little}\ \text{flat},\ \textit{quite}\ \text{round}] \wedge i\,[y_2 = \textit{a little}\ \text{heavy}]$$

$$\omega^s = [y_1 = \textit{quite}\ \text{rounded}] \wedge i\,[y_2 = \textit{very}\ \text{heavy},\ \textit{quite}\ \text{heavy}].$$

(The user has a doubt for ω between *very* and *quite* heavy).

The problem is to know if it is acceptable to say that w belongs to the class of manufactured objects described by a.

Hence q_1^1 (flat) = *a little* ; q_1^1 (rounded) = *quite*; q_2^1 (heavy) = *a little*, r_1^1 (flat) = *nil* ; r_1^1 (rounded) = *quite* ; r_2^1 (heavy) = *very*, r_2^2 (heavy) = *quite*.

A given taxonomy Tax which expresses the background knowledge on the values of M^i makes it possible to say that X (*very, quite*) = *somewhat* ; hence if we settle that

$$r_2^1 \bigcup_i r_2^2(v) = \text{Tax}\ (r_2^1(v), r_2^2(v)) \text{ we have } r_2^1 \bigcup_i r_2^2\ (\text{heavy})$$

$$= \text{Tax}\ (\textit{very, quite}) = \textit{somewhat}.$$

We define L^i by L_1 = not acceptable, L_2 = acceptable, L_3 = completely acceptable and we suppose that the comparison mapping g_i is given by a table T_{gi} such that

$$g_i\left(q_1^1, r_1^1\right) = T_{gi}\ ((\textit{a little}\ \text{flat},\ \textit{quite}\ \text{rounded}), (\textit{nil}\ \text{flat},\ \textit{quite}\ \text{rounded}))$$

$$= \text{acceptable and}$$

$$g_i\left(q_2^1, r_2^1 \bigcup_i r_2^2\right) = T_{gi}\,(\textit{a little}\ \text{heavy},\ \textit{somewhat}\ \text{heavy})$$

$$= \text{not aceptable.}$$

Finally if we settle $f(\{L_i\}) = \text{Min } L_i$ and $L_1 < L_2 < L_3$ we obtain

$$a(\omega) = f_i\left(g_i\left(q_1^1, r_1^1\right), g_i\left(q_2^1, r_2^1 \bigcup_i r_2^2\right)\right) = f_i$$

$$= f_i\ \text{(not acceptable, acceptable)} = \text{not acceptable.}$$

4. Possibilist Objects

4.1 The possibilist approach

Here we follow Dubois and Prade[5] in giving the main idea of this approach.

Definition of a measure of possibility and of necessity

This is a mapping Π from $P(\Omega)$ the power set of Ω in $[0, 1]$ such that

(1) $\Pi(\Omega) = 1\ \Pi(\phi) = 0$

(2) $\forall A, B \subseteq \Omega\quad \Pi(A \bigcup B) = \text{Max}\ (\Pi(A), \Pi(B))$

A measure of necessity is a mapping from $P(\Omega)$ in $[0, 1]$ such that :

(3) $\forall A \subseteq \Omega\ N(A) = 1 - \Pi(\bar{A})$.

The following properties may then be shown :

$N(\phi) = 0; N(A \bigcap B) = \text{Min}$

$(N(A), N(B); \Pi(\bigcup_i A_i) = \text{Max}_i\ (\Pi(A_i)); N(\bigcap_i A_i) = \text{Min}_i$

$(N(A_i)); \Pi(A) \le \Pi(B)$ if $A \subseteq B$; $\text{Max}\ (\Pi(A), \Pi(\bar{A})) = 1$; $\text{Min}\ (N(A), N(\bar{A})) = 0$; $\Pi(A) \ge N(A)$; $N(A) > 0$ implies $\Pi(A) = 1$; $\Pi(A) < 1$ implies $N(A) = 0$; $\Pi(A) + \Pi(\bar{A}) \ge 1$ and $N(A) + N(\bar{A}) \le 1$.

Example — We define $\Pi_E(A)$ (resp. $N_E(A)$) as the possibility (resp. necessity) of getting $\omega \in A$ when $\omega \in E$. We say that $\Pi_E(A) = 1$ if this possibility is true and $\Pi_E(A) = 0$ if not. Hence Π_E and N_E are mappings from $P(\Omega)$ in $\{0, 1\}$. It is then easy to show that Π_E and N_E satisfy the three conditions of their definition.

The theory of possibility models several kinds of semantics, for instance :

i) The physical possibility : this expresses the material difficulty for an action to occur. For instance if I say that : "I have the possibility of carrying 20 kg" ($\Pi(20) = 1$) and I am not able to carry 200 kg ($\Pi(200) = 0$) then I have the possibility of carrying either "20 or 200 kg" ($\Pi(\{20\} \bigcup \{200\}) = \text{Max}\ (\Pi(\{20\}), \Pi(\{200\}) = 1$).

ii. The possibility as a concordance with actual knowledge "it is possible that it will rain or snow today".

iii. The non-astonishment : for instance, "the "typicality" for the color of a flower to be yellow or brown".

4.2 A formal definition of possibilist objects

Here the background knowledge x is denoted p for possibility.

Definition — A possibilist assertion denoted $a_p = \hat{i}\ [y_i = \{q_i^j\ \}_j\ ]$ is an im assertion which takes its values in $L^p = [0, 1]$ such that

- $\forall i\ Q_i$ is a set of measures of possibility.
- OP_p : "$i, q_i^1, q_i^2 \in Q_i\ q_i^1 \bigcup_p q_i^2 = \text{Max}\left(q_i^1, q_i^2\right);\ q_i^1 \bigcap_p q_i^2$ $= \text{Min}\left(q_i^1, q_i^2\right)$; $c_p(q) = 1 - q$ denoted also $\bar{q}$.
- g_p : $g_p\left(q_i^1, q_i^2\right) = \sup\{\min\left(q_i^1(v), q_i^2(v)\right) / v \in O_i\}$
- f_p : $\forall L \subseteq [0, 1] f_p(L) = \text{Min}(\ell / \ell \in L)$

Notice that OP_p is defined as in fuzzy sets and g_p has also been proposed by Zadeh (1971).

It is also possible to define a "necessitist" assertion a_n (thanks to M. O. Menessier, D. Dubois and H. Prades, for their useful remarks which have allowed me to improve this point) by setting :

$$a_n = 1 - \bar{a}_p \text{ where } \bar{a}_p = \hat{i}_p\ [y_i = \bar{q}_i\ ] \text{ and } \bar{q}_i = c_p\,(q_i) = 1 - q_i.$$

This results in $a_n(\omega) = 1 - f(\{g_p(\bar{q}_i, r_i)\}_i)$ and then

$$a_n(\omega) = 1 - \underset{i}{\text{Max}}\ g_p(\bar{q}_i, r_i)$$

$$= 1 - \max\{\sup(\min \bar{q}_i(v), r_i(v)/v \in O_i\}\}_i$$
$$= \min\{1 - \{\sup \min \bar{q}_i(v), r_i(v)/v \in O_i\}\}_i$$
$$= \min\{\inf\{1 - \min \bar{q}_i(v), r_i(v)/v \in O_i\}\}_i$$
$$= \min \inf\{\max(q_i(v), 1 - r_i(v)/v \in O_i\}$$

and then finally $a_n(\omega) = \min g_n(q_i, r_i)$.

It results that a necessitist object is defined by $OP_n = \{\bigcup_n, \bigcap_n, c_n\}$ where $\bigcup_n$ is $\bigcap_p$, $\bigcap_p$ is $\bigcup_p$ and c_n is c_p, $g_n(q_i, r_i) = \inf\{\max(q_i(v), \bar{r}_i(v))/v \in O_i\}$ and $f_n = \min$.

Example — An expert described a class of objects by the following possibilist assertion (restricted, to simplify, to a single event) :

e_p = [height = [around [12, 15], about { 18 }]. An elementary object ω is defined by ω^S = [height = close to 16].

The question is to find the possibility and necessity of ω knowing e_p, in the case where e_p and ω^S may be written :

e_p = [height $= q_1, q_2$] and ω^S = [height $= r_1$] where q_1, q_2, r_1 are possibilist mappings from O = [0, 20] in [0, 1] defined by the background knowledge in figure

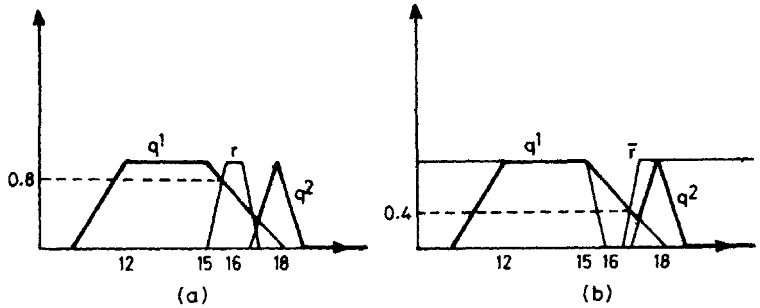

FIG. 1 (a) $q_1 \bigcup q_2$ = Max (q_1, q_2) (b) $\frac{2}{r_i} = 1 - r_i$

1. This means that an object of height 14(resp. 10) has a possibility 1 (resp. $\frac{1}{3}$). It is then possible to compute the possibility of ω by

$$e_p(\omega) = g_p(q_1 \bigcup_p q_2, r_1) = \sup \{\min (q_1 \bigcup_p q_2(v), r_1(v))/v \in \Omega\} = 0.8.$$

The necessity of ω is given by :

$$e_n(\omega) = g_n(q_1 \bigcup_p q_2, r_1) = \inf \{\max (q_1 \bigcup_p q_2(v), r_1(v))/v \in 0\} = 0.4.$$

This example shows that possibilist objects are able to represent not only certitude, variation and doubt but also vagueness (around, about) and inaccuracy (close to 16).

5. Probabilist Objects

5.1. The Probabilist Approach

First we recall the well known axioms of Kolmogorov :

If $C(\Omega)$ is a σ-algebra on Ω (i.e. a set of subsets stable for numerable intersection or union and for complementary). We say that p is a measure of probability on $(\Omega, C(\Omega))$ if

i) $p(\Omega) = 1$

ii) $p(\bigcup_i A_i) = \Sigma p(A_i)$ if $A_i \in C(\Omega)$ and $A_i \bigcap A_j = \phi$.

There are several semantics which follows these axioms : for instance luck in games, frequencies, some kind of uncertainity by subjective probability. Let Q_i be a set of measures of probabilities defined on $(O_i, C(O_i))$.

Definition — A probabilist assertion is an im assertion which takes its values in $L^{pr} = [0, 1]$

O_{pr}: " $q_i^1, q_i^2 \in Q_i\; q_i^1 \bigcup_{pr} q_i^2 = q_i^1 + q_i^2 - q_i^1 q_i^2;\; q_i^1 \bigcap_{pr} q_i^2 = q_i^1 q_i^2$ (which is the mapping which associate to $v \in O_i$, $q_i^1(v)\, q_i^2(v)$;

$$g_{pr} : "\; q_i^1,\; q_i^2 \in O_i\;\; g_{pr}(q_i^1, q_i^2) = < q_i^1, q_i^2 > = \Sigma\, \{q_i^1(v)\, q_i^2(v) / v \in \Omega\}.$$

$f_{pr} : f_{pr}(\{L_i\})$ = mean of the L_i.

To give an intuitive idea of the notion of union of measure of probabilities it is easy to see that if q_i^1 and q_i^2 are the measure of probabilities associated with two dice, $q_i^1 \bigcup_{pr} q_i^2\, (V)$ is the probability that the event V occurs, when the two dice are thrown independently, for one dice or (not exclusive) for the other. Notice that $q_i^1 \bigcup_{pr} q_i^2$ is not a measure of probability because even if $q_i^1 \bigcup_{pr} q_i^2\, (v) \in [0, 1]$ the sum of the $q_i^1 \bigcup_{pr} q_i^2(v)$ on O_i is larger than 1.

5.2. Example

An object ω is described by its color $y_1(\omega)$ which may be red or blue and its roundness $y_2(\omega)$ which may be round or flat.

Let $a = [y_1 = q_i^1, q_i^2]$ pr $[y_2 = q_2]$ and $\omega^S = [y_1 = r_1] \wedge_{pr} [y_2 = r_2]$ where q_1^1 (red) = 0.9;

q_1^1 (blue) = 0.1; q_1^2 (red) = 0.5; q_1^1 (blue) = 0.5; $q2$ (round) = 0.2; $q2$ (flat) = 0.8. It results that a described by two kind of objects: whether often red and rarely blue, or red or blue with equal probability.

By using $q_1^3 = q_1^1 \bigcup_{pr} q_1^2 = q_1^3 + q_1^2 - q_1^1 q_1^2$ we obtain

$$q_1^3 \text{ (red)} = 0.9 + 0.5 - 0.9 \times 0.5 = 0.95$$

$$q_1^3 \text{ (blue)} = 0.1 + 0.5 - 0.1 \times 0.5 = 0.55$$

If r_1 and r_2 are defined as follows :

r_1 (red) = 1, r_1 (blue) = 0; r_2 (round) = 1, r_2 (flat) = 0, it results that

$$a(\omega) = g_{pr}(q_1^3, r_1) \wedge_{pr} g_{pr}(q_2, r_2)$$

$$= (0.95 \times 1 + 0.55 \times 0)\;_{pr}\; (0.2 \times 1 + 0.8 \times 0)$$

$= 0.95 \wedge_{pr} 0.20 = \frac{1}{2}(0.95 + 0.20) = 0.57$, which represents a membership degree for ω to the im object defined by $\boldsymbol{a}$.

6. Belief Objects

6.1. The Belief Function Formalism

The basic notions of this formalism are in Schafer's book[9] : "A mathematical theory of evidence" which is "still a standard reference for this theory" Schafer[10]. First a "probability assignment" function m from $P(\Omega)$ (the power set of Ω, supposed finite) in [0, 1] is defined by : Σ {m (V)/$V \in P(\Omega)$ = 1 and m (*symbof*) = 0; then a belief function bel : $P(\Omega) \rightarrow [0, 1]$ is defined by :

$$\text{Bel } (A) = \Sigma \; \{m \; (V)/V \in P(\Omega), \; V \subseteq A\}$$

A "body of evidence" is veiwed as a pair ($\mathcal{F}$, m) where m is a probability assignment function and $\mathcal{F} = \{V \in P(\Omega)/m\,(V) \neq 0\}$. Given a body of evidence it is possible to define exactly a belief function; it is also possible to define a "plausibilist" function Pl : $P(\Omega) \rightarrow [0, 1]$ such that :

$$\text{Pl } (A) = \Sigma \; \{m \; (V)/V \in P(\Omega), \; V \cap A \neq \phi\}$$

and then we have : Bel $(A) = 1 - $ Pl $(\overline{A})$.
it may be proved (Schafer[9]) that we have the following properties : Bel is a belief function iff :

(i) Bel $(\Omega) = 1$ (ii) Bel $(\phi) = 0$

(iii) $\text{Bel } (A_1 \cup \ldots \cup A_n) \geq \sum_i \text{Bel } (A_i) - \sum_{i<j} \text{Bel } (A_i \cap A_j) + \ldots =$

$$1 \subseteq \sum_{I \neq \phi} \; \{1, ..., n\} \; (-1)^{(I)+1} \text{ Bel } (\bigcap_{i \in I} A_i),$$ where $|I|$ denotes the cardinality of I.

As a consequence of (iii) we get :

$$\text{Pl } (A_1 \cap \ldots \cap A_n) \leq \sum_i \text{Pl } (A_i) - \sum_{i<j} \text{Pl } (A_i \cup A_j) + \ldots$$

Given a belief function Bel, the basic probability assignment function m related to Bel is obtained by :

$$\forall \; A \subseteq P(\Omega) \;\; m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} \text{ Bel } (B).$$

Given two belief functions Bel_1 and Bel_2, their orthogonal sum $\text{Bel}_1 \oplus \text{Bel}_2$, also known as Dempster's rule of combination, is defined by their assotiated probability assignments :

$$m_1 \oplus m_2 (A) = \sum_{V_1 \cap V_2 = A} m_1 (V_1) m_2 (V_2) / \sum_{V_1 \cap V_2 \neq \phi} m_1 (V_1) m_2 (V_2).$$

As a special case, we get a generalization of Bayes rule of conditioning, which is known as Dempster's conditioning :

$$\text{Bel } (A/B) = \frac{\text{Bel } (A \cup \bar{B}) - \text{Bel } (\bar{B})}{(1 - \text{Bel } (\bar{B}))}$$

We have the following link with probability and possibility theories : it may be shown that if $\mathcal{F}$ contains only singletons then Bel is a classical probability measure. Dempster (1967) said that Pl and Bel may be veiwed as upper and lower probabilities. Schafer (1976) has shown that if $\mathcal{F}$ contains only a nested sequence of subsets $V_1 \subseteq V_2 \subseteq \ldots \subseteq V_n$ then we get :

$$\text{Bel } (A \cap B) = \min (\text{Bel } (A), \text{Bel } (B)) \text{ and}$$
$$\text{Pl } (A \cup B) = \max (\text{Pl } (A), \text{Pl } (B))$$

And hence, in this case, Bel and Pl satisfy respectively the properties of necessity and possibility measures. Given a probability measure pr, it may be shown that there exists a possibility, necessity, belief and plausibility function respectively denoted pos, nec, bel, pl, such that nec $\leq$ bel $\leq$ pr $\leq$ pl $\leq$ pos.

The theory of evidence, models several kinds of knowledge :

(i) Probability : as said by J. Pearl (1990) : "belief functions result from assigning [density of] probabilities to sets rather than to individual points".

Example — A machine is able to compute the average number of vehicles whose speeds vary within a set of a priori given intervals for instance V_1 =]0, 110]. Sometimes this machine may fail to give the speed but still be able to give the number of vehicles which pass on the road. If the machine gives for instance the following percentage : 0.40 for speeds which belong in the interval V_1, 0.5 for speeds which belong in V_2 = {speed > 110} and 0.10 when the speed is not known, we may represent this information by a belief function q with body of evidence ($\mathcal{F}$, m) such that

$\mathcal{F} = \{V_1, V_2, IR^+\}$, $m(V_1) = 0.40$, $m(V_2) = 0.50$, $m\ (IR^+) = 0.10$. Then we get for instance bel ([0, 130]) = 0.40 and P_l ([0, 130]) = 0.40 + 0.10 = 0.50.

(ii) Testimony : if two witnesses observe the same event A, then by using the Dempster rule it may be shown that the belief in A increases. if one observes A and the other B with $A \neq B$ and $A \cap B \neq \phi$ then it may be shown the belief in A and B decreases. If $A \cap B = \phi$ the belief in A and B decreases more than in the preceding case and the higher the belief in B, the lower the belief in A.

Example — After an accident observed by two witness, the first one is almost sure that the speed of the vehicule was in the interval V_1 = [0, 100 km] and the second witness who was further away, thinks the same thing but is less sure. Hence, each witness may be represented by a belief function, the first one by q_1, with body of evidence $\{\mathcal{F}_1, m_1\}$ such that

$\mathcal{F}_1 = [V_1 + IR^+]$, $m_1(V_1) = 0.90$ and q_2 defined by $\{\mathcal{F}_2, m_2\}$ such that: $\mathcal{F}_2 = \mathcal{F}_1$ and $m_2(V_1) = 0.70$. Then by using the Dempster rule we get :

$$q_1 \oplus q_2(V_1) = q_1(V_1) + q_2(V_1) - q_1(V_1)\, q_2(V_1) = 0.90 + 0.70 - 0.63 = 0.97.$$

6.2. A formal definition of belief objects

Following Dubois and Prade (1986), we define the union and intersection of two bodies of evidence $(\mathcal{F}_1, m_1)$ and $(\mathcal{F}_2, m_2)$ as follows :

$$\forall\ A = P(\Omega),\ m_1 \bigcup_{bel} m_2(A) = \sum_{V_1 \bigcup V_2 = A} m_1(V_1)\, m_2(V_2);$$

$m_1 \bigcap_{bel} m_2(A) = \sum_{V_1 \bigcap V_2 = A} m_1(V_1)\, m_2(V_2)$ which is consistent with Dempster's rule if the term $m_1 \bigcap m_2(\phi)$ (which reflects the amount of dissonance between the sources or their independence) is eliminated. In the following definition we denote by q_i^j a belief function with body of evidence $(\mathcal{F}_i^j, m_i^j)$.

Definition — A belief assertion denoted $a_{\text{bel}} = \hat{i}_{\text{bel}}\,[y_i = \{q_i^j\}_j]$ is an im assertion which takes its values in L^{bel} = [0, 1], such that :

$\forall i$, Q_i is a set of belief functions defined on O_i

$$.OP_{\text{Bel}} : \forall i,\ q_i^1,\ q_i^2 \in Q_i \quad q_i^1 \bigcup_{bel} q_i^2(V) = \sum_{A \subseteq V} m_i^1 \bigcap_{\text{bel}} m_i^2(A);$$

$q_i^1 \bigcap_{\text{bel}} q_i^2(V) = \sum_{A \subseteq V} m_i^1 \bigcap_{\text{bel}} m_i^2(A)$; the complementary is defined by

$$c_{bel}\left(q_i^j\right)(V) = \overline{q}_i\,(V) = \sum_{A \subseteq V} \overline{m}_i^j\,(A) \text{ where } \overline{m}_i^j(A) = m_i^j\,(\overline{A}).$$

$$g_{bel} : g_{bel}\left(q_i^1,\ q_i^2\right) = \sum \{m_i^1 \bigcap_{bel} m_i^2(V_2) \,/\, V_2 \subseteq V_1, (V_1, V_2) \in \mathcal{F}_1 \times \mathcal{F}_2\}$$

f : the mean.

As in the case of probabilist objects, the choice of the function f may be more general; we have chosen the mean in order to simplify. It is also possible to define a plausibilist object by

$$OP_{p\ell} : q_i^1 \bigcup_{p\ell} q_i^2(V) = \sum_{A \cap V \neq \phi} m_i^1 \cap m_i^2 (A) ; \; q_i^1 \bigcap_{p\ell} q_i^2(V)$$

$$= \sum_{A \cap V \neq \phi} m_i^1 \cup m_i^2 (A)$$

and $c(q_i) = \bar{q}_i$ is defined as in the belief case.

$g_{p\ell} : g_{p\ell}\left(q_i^1, q_i^2\right) = \sum \{m_i^1(V_1)\, m_i^2(V_2) / V_1 \cap V_2 \neq \phi,\ (V_1, V_2) \in \mathcal{F}_1 \times \mathcal{F}_2\}$ and f remains the mean.

The following properties may then be shown : $q_i^1 \bigcup_{b\ell} q_i^2 = q_i^1 q_i^2$;

$$g_{bel}\left(q_i^1, q_i^2\right) = \sum_{V_1 \in \mathcal{F}_1} m_i^1(V_1)\, q_i^2(V_1) \; ;$$

$$g_{p\ell}\left(q_i^1, q_i^2\right) = \sum_{V_2 \in \mathcal{F}_2} m_i^2(V_2)\, P\ell_i^1(V_2) = \sum_{V_1 \in \mathcal{F}_1} m_i^1(V_1)\, pl_i^1(V_2)$$

where $p\ell_i^1(V_j) = \sum_{V \cap V_j \neq \phi} q_i^1(V)$; hence $g_{p\ell}$ is symmetric whereas g_{bel} is not; it is also easy to show that $\forall A \in P(\Omega)$

$$q_i^1 *_{bel}\, q_i^2 (A) = 1 - q_i^1 *_{p\ell}\, q_i^2 (\bar{A}).$$

If two experts observe the same event A and are associated to the belief function q_i^1, q_i^2 with $\mathcal{F}_i^1 = \mathcal{F}_i^2 = \{A, O\}$, then it may be shown that : $q_i^1 \bigcup_{bel} q_i^2 = q_i^1 + q_i^2 - q_i^1 q_i^2 \cdot q_i^2$.

Let us give a simple example.

Example — Several transportation experts define an accident scenario between a car and a bicycle by a belief function q_1 concerning the speed of the car. Knowing q_1 is $\{\mathcal{F}_1, m_1\}$ such that $\mathcal{F}_1 = \{V_1, O\}$, where O is the set of possible speeds and $V_1 \subseteq O$ is an interval of speed (for instance, $V_1 = [100, 120]$ Km/h). Now suppose that a witness observes an accident and says that it is defined by a belief function q_2 with body of evidence $\{\mathcal{F}_1, m_2\}$ such that $\mathcal{F}_2 = \{V_2, O\}$. if we wish to know how much a given accident defined by $\omega^s = [\text{speed} = q_2]$, satisfies the scenario defined by a, we have to compute $a(\omega)$; as a is a belief object, by definition we have :

$$a(\omega) \sum_{V \in \mathcal{F}_1} m_1(V)\, q_2(V)\, q_2(V_1) + m_1(O)\, q_2(O)$$

$a(\omega) = m_1(V_1)\, q_2(V_1) + m_1(O)$. Hence if $V_2 \subseteq V_1\ a(\omega) = m_1(V_1)\, m_2(V_2) + m_1(O)$ and the higher the witness' belief in V_2 the more ω satisfies the scenario defined by a ; if $V_1 \subseteq V_2$ then $a(\omega) = m_1(O)$ and the greater the ignorance of the expert who has defined the scenario, the more ω satisfies the scenario.

7. The Particular Case of Boolean Objects

A boolean object $a = \hat{i}\ [y_i = V_i]$ is an im object $a_b = \hat{i}\ [y_i = q_i]$ where q_i is the characteristic mapping of V_i in O_i, $O\,R_b = \{\bigcup_b, \bigcap_b, c_b\}$ is such that $q_1 \bigcup_b q_2 = \mathrm{Max}(q_1, q_2), q_1 \bigcap_b q_2 = \min(q_1, q_2$ et $c_b(q) = 1 - q$; if $w = \hat{i}\ [y_i = r_i]$ where r_i is the characteristic mapping of $y_i(\omega)$ in $O_i, g_b(q_i, r_i) = < q_i, r_i >$ and $f_b = \min$; it results that if there exists only a single $v \in O_i$ such that $r_i(v) \neq 0$ then $a_b(\omega) = 1$ (thus $r_i \leq q_i$) $\Leftrightarrow a(\omega) =$ true and then $a_b(\omega) = 0 \Leftrightarrow a(\omega) =$ false.

8. Some Qualities and Properties of Symbolic Objects

8.1 Order, Union and Intersection between im Objects

It is possible to define a partial preorder $\leq_\alpha$ on the im objects by getting that: $a_1 \leq_\alpha$ iff $\forall\omega \in \Omega\ \alpha \leq a_1(\omega) \leq a_2(\omega)$.

We deduce from this preorder an equivalence relation R by $A_1\ R\ a_2$ iff Ext $(a_1/\Omega, \alpha) = \mathrm{Ext}(a_2/\Omega, \alpha)$ and a partial order denoted $\leq_\alpha$ and called "symbolic order" on the equivalence classes induced from R.

We say that a_1 inherits from a_2 or that a_2 is more general than a_1, at the level α, iff $a_1 \leq_\alpha a_2$ (which implies $\mathrm{Ext}\alpha\,(a_1/\Omega, \alpha) \subseteq \mathrm{Ext}\alpha\,(a_2/\Omega, \alpha)$).

The symbolic union $a_1 \bigcup_x a_2$ (resp. intersection $a_1 \bigcap_x a_2$) at the level α is the conjunction $\wedge_X$ of the im objects b such that $\mathrm{Ext}\alpha\,(a_1/\Omega, \alpha) \bigcup \mathrm{Ext}\alpha\,(a_2/\Omega, \alpha) \subseteq \mathrm{Ext}(b/\Omega, \alpha)$ (resp. $\mathrm{Ext}\ (\alpha\,(a_1/\Omega, \alpha) \bigcap \mathrm{Ext}\alpha\,(a_2/\Omega, \alpha) \subseteq \mathrm{Ext}\ (b/\Omega, \alpha))$.

8.2 Some Qualities of Symbolic Objects

As in the boolean case, see Brito, Diday[1], it is possible to define different kinds of qualities of symbolic objects (refinement, simplicity, completeness etc.).

For instance, we say that a symbolic object s is complete iff the properties which characterize its extension are exactly those whose conjunction defines the object. More intuitively, if I can see some white dogs and I state "I can see some dogs", my statement doesn't describe the dogs in a complete way since I am not saying that they are white.

On the other hand, the simplicity at level α of an im object is the smallest number of elementary events whose extension at level α coincides with the extension of s at the same level.

8.3 Some properties of im objects : lattice and completness

It may be shown, see Diday[4] for instance, that given a level α the set of im objects is a lattice for the symbolic order and that the symbolic union and intersection define the supremum and infimum of any couple. It may also be shown that the symbolic union and intersection of complete im objects are complete im objects and hence that the set of complete im objects is also a lattice.

9. An Extension of Possibilities, Probabilities and Belief Assertions on Symbolic Objects

Our aim is to extend an im assertion $a = \hat{i}\,[y_i = q_i]$ (where q_i depends on the choice of x and may be for instance a possibility a probability or a belief function), to a dual im assertion denoted a* defined on subsets of a_x the set of im assertions associated to x, and to show that a* is itself a kind of possibility, probability or belief function depending on x. More precisely :

Given $A \subseteq a_x$ we denote $a^*\ell$ a "dual" measure of $a\ell = \hat{i}_x\,[y_i = q_i^\ell]$ and Q_i^A the set of q_i^j such that $a_j = \hat{i}_x\,[y_i = q_i^j] \in A$; then, we settle $a^*\ell(A) = f_x(\{g_x(q_i^\ell, \{\bigcup_{x,j} q_i^j / q_i^j \in Q_i^A\})\}_i)$ and if $* \in \{\bigcup_x, \bigcap_x\}$ then $a^*\ell(A *_x B) = f_x(\{g_x(q_i^\ell, \{ *_{x,j}\, q_i^j / q_i^j \in Q_i^{A \cup B}\})\}_i)$.

Then we have the three following results :

a) In the case of possibilist objects :

Theorem 1 — i) $a^*(a_p) = 1 \quad a^*(\phi) = 0$

ii) $\forall A_1, A_2 \subseteq a_p \quad a^*(A_1 \bigcup_p A_2) = \text{Max}\,(a^*(A_1), a^*(A_2))$.

b) In the case of probabilist objects :

Theorem 2 — i) $a^*(a_{pr}) = 1 \quad a^*(\phi) = 0$

ii) $\forall A_1, A_2 \subseteq a_{pr} \quad a^*(A_1 \bigcup_{pr} A_2) = a^*(A_1) + a^*(A_2) - a^*(A_1 \bigcap_{pr} A_2)$.

c) In case of belief objects :

We say that there is independence between two belief functions q_i^1 and q_i^2 iff $\forall_i$ the bodies of evidence $\left(\mathcal{F}_i^j, m_i^j\right)$ associated to q_i^j for $j = 1, 2$ are such that $m_i^1 \bigcap m_i^2(\phi) = 0$, (or in other words, the focal elements $V_i^1 \in \mathcal{F}_i^1$, $V_i^2 \in \mathcal{F}_i^2$ are such that : $V_i^1 \bigcap V_i^2 \neq \phi$). Two subsets A_1, A_2 of a_{bel} are said independent iff for $\forall_i$ and $\ell = 1, 2$ such that $Q_i^\ell = \bigcup_{bel} \{q_i^\ell / q_i^\ell \in Q_i^{A\ell}\}$, Q_i^1 and Q_i^2 are independent.

Theorem 3.— i) $a^*(a_{bel}) = 1, \; a^*(\phi) = 0$

ii) If $\forall_i\, A_i \subseteq a_{bel}$ are independent then :

$$a^*\Big(\bigcup_{bel,\; i \in \{1,\dots,n\}} A_i\Big) \geq \sum_{I \subseteq \{1,\dots,n\}} (-1)^{|I|+1}\, a^*\Big(\bigcap_{bel,\; i \in I} A_i\Big)$$

iii) If $\forall_i\ A_i \subseteq a_{bel}\ m^*(A) = \dfrac{a^*_{bel}(A)}{a^*_{bel}(h(A))} \sum_{B \subseteq A} (-1)^{|A-B|}\ a^*_{bel}(h(B))$

where $h(B) = \bigcap_{bel} \{A_i / A_i = A - \{a_i\},\ a_i \in A\ B,\ B \neq A\}$

$$h(A) = \bigcup_{bel} \{A_i / A_i = A - \{a_i\},\ a_i \in A\}$$

then m^* is a probability assignment function on a_{bel} (in other works : $m^* : P(a_{bel}) \to [0,\ 1]$ is such that $m^*(\phi) = 0$, $\sum_{A \subseteq a_{bel}} m^*(A) = 1$ and "$A \subseteq a_{bel}\ a^*(A) = \sum_{B \subseteq A} m^*(B)$).

By using m^* it is then possible to extend Dempster's rule and Dempster's conditioning on the set of belief assertions.

Semantic of a^* in case of belief objects

The meaning of $a_1^*(a_2)$ may be interpreted as a "belief of belief" or the "conviction" of someone denoted E_1, whose belief is represented by a_1, of the belief of someone else, denoted E_2, whose belief is represented by a_2.

Example — For $i = 1, 2$, let be $a_i = [y = q_i]$ whereq_i is a belief function $O \to [0,\ 1]$ with body of evidence $(\mathcal{F}_i, m_i)$ and $\mathcal{F}_1 = \mathcal{F}_2 = \{A,\ B,\ O\}$ with $A \bigcap B = \phi$; then we get

$$a_1^*(a_2) = g_{bel}(q_1, q_2) = \sum_{V \in \mathcal{F}_1} m_1(V)\, q_2(V) = m_1(A)\, m_2(A)$$

$$+ m_1(B)\, m_2(B) + m_1(O).$$

Hence, we can see that the conviction of E_1 concerning the belief of E_2 will be maximum (i.e. $a_1^*(a_2) = 1$) if a_1 is totally ignorant of the evidence A and B (because in that case $m_1(A) = m_1(B) = 0$ and $m_1(O) = 1$) and if E_1 and E_2 totally believe the same evidence (because $m_1(A) = m_2(A) = 1$ or $m_1(B) = m_2(B) = 1$). If $m_1(B) = 0$ and E_1 and some ignorance of A (i.e. $m_1(O) \in]\ 0,\ 1\ [$) then his conviction of the belief of E_2 on A (i.e. $q_2(A)$) will be greater than $q_2(A)$ (for instance if $m_1(A) = m_2(A) = \frac{1}{2}$ then $m_1(O) = \frac{1}{2}$ and the conviction of E_1 will be $a_1^*(a_2) = 0.75$). If E_1 totally believes $A(m_1(A) = 1, m_1(B) = m_1(O) = 0)$ and E_2 totally believes B $(m_2(B) = 1, m_1(A) = 0)$ then, the conviction of E_1 of the belief of E_2 will be 0. If E_2 is totally ignorant (i.e. $m_1(A) = m_2(B) = 0$) then the conviction of E_1 of the belief of E_2 will be low if his belief is strong (i.e. his ignorance measured by $m_1(O)$ is low).

There is an analogous theorem if a_1 is a plausibilist assertion and $a_1^*(a_2)$ may be interpreted as the mutual "non-discordance" between what E_1 and E_2 believe. To illustrate that, going back at the preceding example we can see that if a_1 is a plausibilist object then :

$$a_1^*(a_2) = g_{pl}(q_1, q_2) = \sum_{V \in \mathcal{F}_1} m_1(V)\, pl_2(V) = m_1(A)\,(m_2(A) + m_2(O)) + m_1(B)$$

$$(m_2(B) + m_2(O)) + m_1(O)\, pl_2(O) = m_1(A)\, m_2(A) + m_1(B)\, m_2(B)$$
$$+ m_1(O) + m_2(O) - m_1(O)\, m_2(O).$$

Hence, this corresponds to intuition as we can see (contrary to the case of conviction) that the non-discordance between what E_1 and E_2 believe remains high when E_2 is totally ignorant (i.e. $m_1(A) = m_2(B) = 0$) even if the belief of E_1 is strong (i.e. $m_1(O) = 0$).

Another kind of interpretation of $a_1^*(a_2)$ may be obtained in terms of "fitness"; if we consider the class C_i (of fruits produced by a village, for instance) described the belief object a_i, we may say, when a_1 is a belief object, that $a_1^*(a_2)$ measures how much C_2 "fit" C_1; when a_1 is a plausibilist object, we may say that $a_1^*(a_2)$ measures the "non-disagreement" between C_1 and C_2. For instance, if y expresses the color and the fruits of both villages have the same color, denoted A, (i.e. $m_1(A) = m_2(A) = 1$, $m_1(B) = m_2(B) = 0$, $m_1(O) = m_2(O) = 0$) then $a_1^*(a_2) = 1$ measures how much C_2 "fit" C_1 and also the "non-disagreement", for the color, between C_1 and C_2. If the color of the fruits of the second village is A (i.e. $m_1(A) = 1$, $m_1(O) = 0$) then, when a_1 is a belief object, we get $a_1^*(a_2) = 0$ which measures how much C_2 fit C_1; when a_1 is plausibilist object, we get $a_1^*(a_2) = 1$ which measures the non-disagreement between C_1 and C_2.

10. Statistic and Data Analysis of Symbolic Objects

Several studies have recently been carried out in this field : for histograms of symbolic objects, see De Carvalho & al (1990) and (1991); for generating rules by decision graph on im objects in the case of possibilist objects with typicalities as modes see Lebbe and Vignes[7]; for generating overlapping clusters by pyramids on symbolic objects see Brito, Diday[1].

More generally, four kinds of data analysis may roughly be defined depending on the imput and output : *a*) numerical analysis of classical data tables *b*) numerical analysis of symbolic objects (for instance by defining distances between objects Gowda, Diday[2]) *c*) symbolic analysis of classical data tables, for instance obtaining a factor analysis or a clustering automatically interpreted by symbolic objects *d*) symbolic analysis of symbolic objects where the input and output of the methods are symbolic objects.

CONCLUSION

Starting from classes of individual objects defined by intention (in contrast to classes only defined by the set of their members) we have given several ways to define them by a mapping denoted a_x on Ω (the set of individual objects) depending on the background knowledge x. A question naturally arose : is it possible to say that a_x (ω) measures a "probability", "possibility" or a "belief" that ω belongs to the class represented by a_x? To answer this question we needed to extend a_x and a_x^* defined on a_x a set of symbolic objects and to define set operators OP_x = $\{\bigcup_x, \bigcap_x, c_x\}$ in a_x adapted to x. if we say that classical sets represent a knowledge level of order 0; probability, possibility and belief, a knowledge level of order 1, the question was now to know if a_x^* represents a knowledge level of order 2. In other words, if it is a kind of probability of probability, possibility of possibility, belief of belief respectively associated with the corresponding operators OP_x; the theorems 1, 2, 3 show that it is the case, if OP_x and the functions g and f are well chosen.

Unlike most work carried out in Artificial Intelligence, symbolic data analysis constitutes a "critique of pure reasoning" by giving less importance to the reasoning and more importance to the statistical study of knowledge bases, considered as a set of "symbolic objects".

A wide field to research is opened by extending classical statistics to statistics of intentions and more specially by extending problems, methods and algorithms of data analysis and pattern recognition to symbolic objects.

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