Indian J. pure appl. Math., **25**(5); 513-518, May 1994

PERTURBATIONS OF NONLINEAR SYSTEM OF DIFFERENCE EQUATIONS

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(Received 22 *July* 1991*; after final revision* 11 *November* 1993*; accepted* 26 *November* 1993*)*

This paper deals with the asymptotic behaviour of solutions of nonlinear difference systems and its perturbed systems. The main tools used here are discrete generalization of Gronwall-Bellman inequality and a nonlinear variation of constants formula.

1. INTRODUCTION

One of the most useful methods available for studying the asymptotic behaviour of solutions of a nonlinear system of differential equations involved comparsion with a suitable linear system and the use of the variation of constants formula. In the discrete theory an analogues of variation of constants formula has been successful in considering properties of perturbed linear difference equations[4-6]. The purpose of this paper is to study the asymptotic behaviour of solutions of nonlinear difference systems and its perturbed systems by means of an analogue of nonlinear variation of constants for difference equations given in Lord[3].

2. PRELIMINARIES

In this section we define various notations and terms which will be used in our subsequent discussion. Let $N_{n_0} = \{n_0, n_0+1, \ldots\}$, n_0 a non-negative integer. Consider now the difference system

$$x_{n+1} = f(n, x_n)\,,\ x_{n_0} = x_0 \qquad \ldots (1)$$

and its perturbed system

$$y_{n+1} = f(n, y_n) + F(n, y_n)\,,\ y_{n_0} = x_0 \qquad \ldots (2)$$

where $f, F : N_{n_0} \times R^s \to R^s$ and $\dfrac{\partial f}{\partial x}$ exist and be continuous and invertible on $N_{n_0} \times R^s$.

Our first preliminary result is a nonlinear variation of constants given in Lord[3].

Lemma 2.1 — Assume (1) admits solutions $x(n, n_0, x_0)$ for $n \geq n_0$ and $\psi(n, n_0, y_0, x_0)$ is invertible for all $n \geq n_0$. If v_n is a solution of

$$v_n = x_0 + \sum_{j=n_0}^{n-1} \psi^{-1}(j+1, n_0, v_j, v_{j+1}) F(j, v_j) \qquad \text{... (3)}$$

then any solution $y(n, n_0, x_0)$ of (2) satisfies the relation

$$y(n, n_0, x_0) = x(n, n_0, v_n) \qquad \text{... (4)}$$

where

$$\psi(n, n_0, v_j, v_{j+1}) = \int_0^1 \phi(n, n_0, t v_j + (1-t) v_{j+1})\, dt$$

and

$$\phi(n, n_0, x_0) = \frac{\partial x(n, n_0, x_0)}{\partial x_0}$$

is the matrix valued solution of the variational equation

$$\phi(n+1, n_0, x_0) = H(n, n_0, x_0)\, \phi(n, n_0, x_0)$$

$$\phi(n_0, n_0, x_0) = I$$

where $$H(n, n_0, x_0) = \frac{\partial f(n, x(n, n_0, x_0))}{\partial x}$$

and I is the identity matrix.

The following example is given to demonstrate the computability of ψ and ψ^{-1}.

Consider the scalar equation

$$x_{n+1} = x_n/[1 + x_n]$$

$$x_{n_0} = x_0, x_0 \in [0, \infty). \qquad \text{... (5)}$$

The solution of (5) is given by

$$x_n = x_0/[1 + x_0(n - n_0)], \quad n \in N_{n_0}$$

$$\phi(n, n_0, x_0) = \frac{1}{[1 + x_0(n - n_0)]^2}$$

and an easy calculation yields

$$\psi(n, n_0, v_j, v_{j+1}) = 1/[1 + v_j(n - n_0)]\,[1 + v_{j+1}(n - n_0)]$$

$$\psi^{-1}(n, n_0, v_j, v_{j+1}) = [1 + v_j(n - n_0)]\,[1 + v_{j+1}(n - n_0)].$$

The second preliminary result is a discrete generalization of Gronwall-Bellman inequality

Lemma 2.2 — If the following inequality

$$m(n) \le m(n_0) + \sum_{s=n_0}^{n-1} [a_s m^\alpha(s) + b_{s+1} m(s+1)], \quad n \in N_{n_0} \qquad \text{... (6)}$$

where $m, a : N_{n_0} \to R^+$, $b : N_{n_0} \to [0, 1)$ and $0 < \alpha < 1$ are satisfied.

Then for $n \in N_{n_0}$

$$m(n) \le m(n_0) E(n) [1 + (1-\alpha) Q(n)]^{1/1-\alpha} \qquad \text{... (7)}$$

where $E(n) = \exp \sum_{s=n_0+1}^{n} \left(\frac{b_s}{1-b_s} \right)$

and $Q(n) = m^{\alpha-1}(n_0) \sum_{s=n_0}^{n-1} a_s E_s^\alpha$.

Proof : We can rewrite the given inequality as

$$m(n)(1-b_n) \le L(n) + \sum_{s=n_0+1}^{n-1} b_s m(s), \quad n \in N_{n_0}$$

$$L(n) = m(n_0) + \sum_{s=n_0}^{n-1} a_s m^\alpha(s).$$

Since $m(n)$ and a_n are non-negative so that $L(n)$ is nondecreasing, and if there we have $m(n_0) > 0$ then we obtain from the above inequality

$$m(n) \le \left[\frac{L(n)}{1-b_n} \right] \underset{s=n_0+1}{\overset{n-1}{\pi}} \left(1 + \frac{b_s}{(1-b_s)} \right).$$

Since $\frac{1}{1-b_n} = 1 + \frac{b_n}{1-b_n}$, we have from the above inequality

$$m(n) \le L(n) \underset{s=n_0+1}{\overset{n}{\pi}} \left(1 + \frac{b_s}{(1-b_s)} \right)$$

$$\le L(n) E(n) \qquad \text{... (8)}$$

and hence we have

$$m(n) \le E(n) [m(n_0) + \sum_{\delta=n_0}^{n-1} a_s m^\alpha(s)], \quad n \in N_{n_0}.$$

From the definition of $L(n)$ we have

$\Delta L(n) = a_n m^\alpha(n) \le a_n E^\alpha(n) L^\alpha(n)$ for $n \in N_{n_0}$. ... (9)

Since $\int_{L(n)}^{L(n+1)} dz/z^{\alpha} \leq \frac{\Delta L(n)}{L^{\alpha}(n)}$

we have from (9)

$$\int_{L(n)}^{L(n+1)} dz/z^{\alpha} \leq a_n E^{\alpha}(n) \text{ for } n \in N_{n_0}.$$

Summing the last inequality from n_0 to $n - 1$, we obtain

$$L(n) \leq m(n_0)\,[1 + (1-\alpha)\,Q(n)]^{1/1-\alpha}$$

and now the desired estimate (7) follows from (8).

If $m(n_0) = O$, we may put $m(n_0) + \varepsilon$ instead of $m(n_0)$ and then let $\varepsilon \to 0$ in the obtained estimate to complete the proof.

Remark : For different types of discrete generalizations of Gronwall-Bellman inequality see Agarwal and Thandapani[1] and the references cited therein.

Finally we state some definitions given in Dannan and Elaydi[2] and Pachpatte[4].

Definition 2.3 — The zero solution is said to be exponentially asymptotically stable for (1) if there exists $m_1 > 0$, $c > 0$ such that for any solution $x(n, n_0, x_0)$ there is

$$||\, x(n, n_0, x_0)\,|| \leq c\,||\, x_0\,||\, e^{-m_1(n-n_0)}, \quad n \in N_{n_0}.$$

Definition 2.4 — The zero solution is said to be uniformly Lipschitz stable for (1) if there exists $m_2 > 0$ and $\partial > 0$ such that for any solution $x(n, n_0, x_0)$ there is

$$||\, x(n, n_0, x_0)\,|| \leq m_2\,||\, x_0\,||, \text{ for } n \in N_{n_0} \text{ and } ||\, x_0\,|| < \partial.$$

Definition 2.5 — The zero solution is said to be globally uniformly Lipschitz stable for (1) if there exists $m_3 > 0$ such that for any solution $x(n, n_0, x_0)$ there is

$$||\, x(n, n_0, x_0)\,|| \leq m_3\,||\, x_0\,||, \text{ for } n \in N_{n_0} \text{ and } ||\, x_0\,|| < \infty.$$

Definition 2.6 — A solution $x(n, n_0, x_0)$ of (1) is said to be slowly growing if there exists $m_4 > 0$ and $\varepsilon > 0$ such that

$$||\, x(n, n_0, x_0)\,|| \leq m_4\,||\, x_0\,||\, e^{\varepsilon(n-n_0)}, \quad n \in N_{n_0}.$$

3. Main Results

Theorem 3.1 — Suppose the solution of (2) admits a representation (4) for all $n \in N_{n_0}$ where v_n is given by (3). Suppose the zero solution of (1) is exponentially asymptotically stable with exponent m. Further assume

(i) $||\, \psi^{-1}(n, n_0, y_0, x_0)\, F(n, x(n, n_0, x_0))\,|| \leq a_n\,||\, y_0\,||^{\alpha} + b_{n+1}\,||\, x_0\,||$

where a_n, b_n and α are as defined in Lemma 2.2.

(ii) $\sum_{s=n_0+1}^{n} \frac{b_s}{1-b_s} < \partial (n-n_0)$ and $\partial < m$

(iii) $\sum_{s=n_0}^{n-1} a_s \exp\left(\alpha \sum_{\partial=n_0+1}^{s} \frac{b_\sigma}{1-b_\sigma}\right) < \infty$ for all $n \in N_{n_0}$.

Then the zero solution of (2) is exponentially asymptotically stable.

PROOF : From (3) and (i), letting $m(n) = \| v_n \|$ we have

$$m(n) \le m(n_0) + \sum_{j=n_0}^{n-1} \| \psi^{-1}(j+1, n_0, v_j, v_{j+1}) F(j, x(j, n_0, v_j)) \|$$

$$\le m(n_0) + \sum_{j=n_0}^{n-1} a_j m^\alpha(j) + b_{j+1} m(j+1).$$

Now by Lemma 2.2 and (ii) we obtain

$m(n) \le K(n) m(n_0) \exp(\partial(n-n_0))$

where

$K(n) = [1 + (1-\alpha) Q(n)]^{1/1-\alpha}, \quad n \in N_{n_0}$.

By (iii), there exists a constant $K_1 > 0$ such that

$$K(n) \le K_1 \text{ for all } n \in N_{n_0}.$$

Thus by (4) and the exponential asymptotic stability of zero solution of (1) implies

$$\| y(n, n_0, x_0) \| = \| x(n, n_0, v_n) \| \le c \| v_n \| e^{-m(n-n_0)}$$

$$\le K_1 c \| x_0 \| e^{-(m-\partial)(n-n_0)}.$$

The exponential asymptotic stability of the zero solution of (2) follows since $m - \partial > 0$.

Theorem 3.2 — Let all hypotheses of Theorem 3.1 are satisified except the condition (ii) replaced by

(ii)$'$ $\sum_{s=n_0+1}^{\infty} \frac{b_s}{1-b_s} < \infty$.

If the zero solution of (1) is uniformly Lipschitz stable (globally uniformly Lipschitz stable) then the zero solution of (2) is uniformly Lipschitz stable (globally uniformly Lipschitz stable).

PROOF : From (3) and (i) letting $m(n) = \| v_n \|$ we have

$$m(n) \le m(n_0) + \sum_{j=n_0}^{n-1} a_j m^{\alpha}(j) + b_{j+1} m(j+1).$$

Now by Lemma 2.2 and (ii) and (iii) we obtain

$$m(n) \le K_1 m(n_0)$$

where K_1 is some constant. Thus by (4) and the uniformly Lipschitz stability of zero solution of (1) implies

$$\| y(n, n_0, x_0) \| = \| x(n, n_0, v_n) \| \le m_2 \| v_n \| \le K_1 m_2 \| x_0 \|.$$

The uniformly Lipschitz stability of zero solution of (2) follows since $\| x_0 \| < \delta$. Similarly we prove the globally uniformly Lipschitz stability of zero solution of (1) implies that of (2).

Theorem 3.3 — Let all hypotheses of Theorem 3.1 are satisfied except the condition (ii) replaced by

$$\text{(ii)} \quad {}'' \sum_{s=n_0+1}^{n} \frac{b_s}{1-b_s} < \varepsilon (n - n_0).$$

If every solution of (1) is slowly growing then every solution of (2) is slowly growing.

PROOF : Since the proof can be modelled as that of Theorem 3.1 and hence the details are omitted.

Several other notions of stability for the perturbed equation (2) by using variation of constants will form our forthcoming research paper.

ACKNOWLEDGEMENT

The author is thankful to the referee for suggesting several improvements.

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