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CONTIGUOUS RELATIONS FOR THE *G*-FUNCTIONS OF TWO VARIABLES

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Contiguous relations for various hypergeometric types of functions go back as fas as Gauss. Recently these ideas were extended to functions defined by Mellin-Barnes integrals. Extensions have been obtained for the hypergeometric functions of two and more variabies, as well as extensions to multiple Mellin-Barnes integrals, but most of these results were for constant coefficient recurrences. Extensions are now explored for the contiguous relations with linear coefficients.

1. INTRODUCTION

The terminology contiguous function will be used in the same manner as for hypergeometric functions to mean that exactly one of the parameters is shifted by ± 1. For example, the function $F(a + 1, b; c; z)$ is contiguous to $F(a, b; c; z)$. In this sense, relations among such functions provide fundamental recurrence relations from which others can be obtained. For the Gauss function the list of contiguous function relations can be found in Erdélyi *et al.*[9], Luke[13], or Slater[21]; for the Kummer function, in Slater[20]. Analogs for the Appell functions are given in Appell and Kampé de Fériet[1] and Buschman[4]. The generalized hypergeometric function ${}_pF_q$ was treated by Rainville[18,19]. Some recurrences for Lauricella functions are listed by Exton[11]. For Meijer's *G*-function and Fox's *H*-function some relations can be found in Erdélyi *et al.*[9], and expanded results appear in Buschman[2,5]. For some extensions to two variables, see Buschman and Gupta[8] or Srivastava *et al.*[22]. Some general theorems which produce contiguous relations with constant coefficients (named simple relations by Rainville) have been given by Buschman[6,7]. The theorems apply to those functions represented by power series or by Mellin-Barnes integrals which have at least some hypergeometric properties in their coefficients or in their integrands.

However, the contiguous relations with linear coefficients (named less simple by Rainville) involve some complications, which can be seen already in the work of

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Rainville[18,19] and later in Buschman[4]. Some of these complications become worse with the increase in the number of variables. In this work we study such less simple contiguous relations. Since we cannot expect to extend the results to the H-functions of Fox[12], see the comments at the end of Buschman[5], we concentrate on the linear coefficient contiguous function relations for the G-function of two variables.

The principle tool in this work will be the two-dimensional Mellin transformation,

$$\mathbf{M}\{F(x, y)\} = \int_0^\infty \int_0^\infty F(x, y)\, x^{s-1} y^{t-1}\, dx\, dy. \qquad \ldots (1.1)$$

Although good tables of one-dimensional transformations exist, Erdélyi *et al.*[10], none exist for two-dimensional Mellin transformations. (The Laplace table of Voelker and Doetsch[23] is not very useful here, although some of their theorem give ideas for the kinds of results that might be available). We try to pattern our work after that in Buschman[3]. Consequently, we need the analogs for some transforms, which we list in Section 3. But first we explore the earlier work in order to obtain the simple relations. The amount of computation required for the investigation of less simple relations can be reduced by use of the simple relations in order to delete redundant terms in advance. Throughout this work whenever we write a formula we assume sufficient conditions such that each of the G-functions is defined and that each G-function has a sufficient number of parameters for the formulas to make sense. These assumptions will save much repeated writing of the conditions of validity.

Properties of the G-function can be found in Erdélyi *et al.*[10], Mathai and Saxena[14], and Srivastava, Gupta and Goyal[22]. We take the definition of, and the notations for, the G-function of two variables in the form

$$G^{m_1, n_1 : m_2, n_2 : m_3, n_3}_{p_1, q_1 : p_2, q_2 : p_3, q_3}\left[\begin{array}{c|c} x & (a):(c);(e) \\ y & (b):(d);(f) \end{array}\right]$$

$$= \frac{1}{-4\pi^2} \int_{L_1}\int_{L_2} \Phi(s, t)\, \Theta_1(s)\, \Theta_2(t)\, x^{s-1} y^{t-1}\, ds\, dt \qquad \ldots (1.2)$$

where, for example, (a) denotes a set of parameters and where

$$\Phi(s, t) = \frac{\prod_{j=1}^{n_1} \Gamma(1 - a_j + s + t) \prod_{j=1}^{m_1} \Gamma(b_j - s - t)}{\prod_{j=m_1+1}^{q_1} \Gamma(1 - b_j + s + t) \prod_{j=n_1+1}^{p_1} \Gamma(a_j - s - t)} \qquad \ldots (1.3)$$

and

$$\Phi_1(s) = \frac{\prod_{j=1}^{m_2} \Gamma(d_j - s) \prod_{j=1}^{n_2} \Gamma(1 - c_j + s)}{\prod_{j=m_2+1}^{q_2} \Gamma(1 - d_j + s) \prod_{j=n_2+1}^{p_2} \Gamma(c_j - s)} \qquad \ldots (1.4)$$

with the third factor of the integrand defined analogously in terms of the parameters

e and f. The convergence conditions of Buschman[3] can easily be modified to cover this form of the definition.

For convenience, we assume throughout that no pair of the parameters are equal or differ by 1. Although some additional complications do appear in such cases, basically the same methods can be used.

2. From Earlier Work

It is common notation to suppress all parameters except those involved in the contiguous functions and we do that here. Some of the simple contiguous relations appear in, or can be obtained directly from our earlier work. For example, since the parameters c and d appear in exactly the same manner in the G-function of two variables as do the parameters for the G-function of one variable, we immediately can write the three recurrences

$$G[c_1-1]+G[d_1+1]=(d_1-c_1+1)\,G \qquad \text{... (2.1)}$$

$$G[c_1-1]+G[c_p-1]=(c_p-c_1)\,G \qquad \text{... (2.2)}$$

$$G[c_1-1]-G[d_q+1]=(d_q-c_1+1)G. \qquad \text{... (2.3)}$$

We establish the convention here that, because of the symmetries, the subscript 1 implies a parameter from the first (numerator) subset and the subscript p or q a parameter from the second (denominator) subset. From symmetry we also have the analogous set of three formulas in which c and d are replaced throughout, respectively, by e and f. Such formulas were obtained for the H-function by Buschman and Gupta[8]. By a further analogous development we obtain three formulas in which c and d are replaced by a and b. These nine formulas allow us to concentrate later on the parameters a_1, c_1 and e_1, since we can now use these formulas to easily shift from the three parameters to other parameters.

Some 4-term relations were also obtained by Buschman and Gupta[8]. One example is

$$G[a_1-1]-G[c_1-1]-G[e_1-1]=(c_1+e_1-a_1-1)G. \qquad \text{... (2.4)}$$

Other parameters can now be introduced as in the 3-term relations in order to obtain many other contiguous relations.

A characteristic of the simple recurrences is that the parameters from three of the sets are always increased and those from the other three sets are always decreased. We need the less simple recurrences in order to introduce any parameter shifts in the other directions.

A careful inspection of the derivations in Buschman[5] provides some ideas for the construction of less simple contiguous relations in the two variable cases, but direct reinterpretation does not work because of the interaction between sets of parameters. The problems involving the less simple relations are much more complicated and they are discussed in Section 4.

3. Notations and Formulas

In order to examine the less simple relations we need to establish some further notations and formulas. As in Buschman[4,5] we write out some Mellin transforms. Some of these can be written down by merely inspecting the earlier work; others need to be computed. First, we denote the two-dimensional Mellin transform of our G-function by

$$\mathbf{M}\left\{ G^{m_1, n_1 : m_2, n_2 : m_3, n_3}_{p_1, q_1 : p_2, q_2 : p_3, q_3} \left[\begin{array}{c|c} x & (a):(c);(e) \\ y & (b):(d);(f) \end{array} \right] \right\}$$
$$= \Gamma\left[\begin{array}{l} (1-a-s-t), (b+s+t), (1-c-s), (d+s), (1-e-t), (f+t) \\ (a+s+t), (1-b-s-t), (c+s), (1-d-s), (e+t), (1-f-t) \end{array} \right] \quad \text{... (3.1)}$$

in which we use the notation of Slater[21] for the Γ-function. In order to provide simplifications later let

$$(-1)^{n_1+n_2+n_3}\, \Omega(s,t)$$
$$= \Gamma\left[\begin{array}{l} (-a-s-t), (b+s+t), (-c-s), (d+s), (-e-t), (f+t) \\ (1+a+s+t), (1-b-s-t), (1+c+s), (1-d-s), (1+e+t), (1-f-t) \end{array} \right] \quad \text{... (3.2)}$$

which we use in the following Mellin transforms.

$$\mathbf{M}\{G\} = \prod^{p_1} (a+s+t) \prod^{p_2} (c+s) \prod^{p_3} (e+t)\ \Omega\, (-1)^{n_1+n_2+n_3} \quad \text{... (3.3)}$$

$$\mathbf{M}\{xG\} = \prod^{q_1} (b+s+t) \prod^{q_2} (d+s) \prod^{p_3} (e+t)\ \Omega\, (-1)^{q_1-m_1+q_2-m_2+n_3} \quad \text{... (3.4)}$$

$$\mathbf{M}\{yG\} = \prod^{q_1} (b+s+t) \prod^{p_2} (c+s) \prod^{p_3} (f+t)\ \Omega\, (-1)^{q_1-m_1+q_3-m_3+n_2} \quad \text{... (3.5)}$$

$$\mathbf{M}\{G[a_1+1]\} = \prod_{2}^{p_1} (a+s+t) \prod^{p_2} (c+s) \prod^{p_3} (e+t)\ \Omega\, (-1)^{n_1+n_2+n_3-1} \quad \text{... (3.6)}$$

with analogs for $G[c_1 + 1]$ and $G[e_1 + 1]$.

$$\mathbf{M}\{G[a_1-1]\} = (a_1-1+s+t) \prod^{p_1} (a+s+t) \prod^{p_2} (c+s) \prod^{p_3} (e+t)$$
$$\Omega\, (-1)^{n_1+n_2+n_3+1} \quad \text{... (3.7)}$$

with analogs for $G[c_1 - 1]$ and $G[e_1 - 1]$ here and in the next two formulas.

$$\mathbf{M}\{xG[a_1-1]\} = (a_1+s+t)\prod^{q_1}(b+s+t)\prod^{q_2}(d+s)$$
$$\prod^{p_3}(e+t)\ \Omega\,(-1)^{q_1-m_1+q_2-m_2+n_3+1}, \quad \text{... (3.8)}$$

$$\mathbf{M}\{yG[a_1-1]\} = (a_1+s+t)\prod^{q_1}(b+s+t)\prod^{p_2}(c+s)$$
$$\prod^{q_3}(f+t)\ \Omega\,(-1)^{q_1-m_1+q_3-m_3+n_2+1}. \quad \text{... (3.9)}$$

The next two pairs have analogs involving d and f, respectively.

$$\mathbf{M}\{xG[b_1-1]\} = \prod_2^{q_1}(b+s+t)\prod^{q_2}(d+s)\prod^{p_3}(e+t)$$
$$\Omega\,(-1)^{q_1-m_1+q_2-m_2+n_3} \quad \text{... (3.10)}$$

$$\mathbf{M}\{xG[b_q-1]\} = \prod^{q_1-1}(b+s+t)\prod^{q_2}(d+s)\prod^{p_3}(e+t)$$
$$\Omega\,(-1)^{q_1-m_1+q_2-m_2+n_3-1} \quad \text{... (3.11)}$$

$$\mathbf{M}\{yG[b_1-1]\} = \prod_2^{q_1}(b+s+t)\prod^{p_2}(c+s)\prod^{q_3}(f+t)$$
$$\Omega\,(-1)^{q_1-m_1+q_3-m_3+n_2}, \quad \text{... (3.12)}$$

$$\mathbf{M}\{yG[b_q-1]\} = \prod^{q_1-1}(b+s+t)\prod^{p_2}(c+s)\prod^{q_3}(f+t)$$
$$\Omega\,(-1)^{q_1-m_1+q_3-m_3+n_2-1}. \quad \text{... (3.13)}$$

Throughout the above formulas we have displayed the lower index on the product only when it is not equal to 1. We have suppressed the indices of summation for further condensation of notation. Formulas which involve the subscripts 1 and q on the parameters again can readily be adjusted for any elements from the first and second subsets, respectively. Others are not listed because they lead to redundancies in view of the simple relations of Section 2.

4. Less Simple Relations

In order to determine information on the coefficients in the system of equations, we require that the polynomial equation in the two variables s and t be an identity. It now becomes important to count the number of terms, N_E, in the product expansions such as

$$\prod^{q_1}(b+s+t)\prod^{q_3}(d+s)\prod^{q_3}(f+t), \quad \text{... (4.1)}$$

in order to determine the number of equations that our undetermined coefficients must satisfy. These terms can easily be counted, if we note first that the expansion of the first factor generates $(q_1+1)(q_1+2)/2$ terms which can be arranged in a triangular array according to degree and with $q_1 + 1$ terms on a side. A geometric argument about lattice points now helps. We note that multiplication by $d + s$ sweeps this triangle in the same direction as one side. Next, multiplication by $f + t$ sweeps the entire previously generating figure in the direction of the other side. The final general figure swept out is a parallelogram of lattice points with $q_1 + 1 + q_2$ on one side and $q_1 + 1 + q_3$ on the other, except that a triangle with q_1 points on a side was not swept out. Hence the number of terms is

$$N_E = (q_1+q_2+1)(q_1+q_3+1) - q_1(q_1+1)/2. \qquad \text{... (4.2)}$$

The number of independent equations is, however, not easy to determine.

We note that the count of the equations to be satisfied is a quadratic function, whereas the number of unknowns is only a linear function. Consequently, as such parameters increase in number, the system of equations may well be expected to have only the trivial solution. Hence it now should not be surprising that there may be no contiguous relations of the less simple type with linear coefficients; that is, of the type that we are seeking. An increase in the number of unknowns by moving to higher degree polynomial coefficients also has shown no promise.

Consequently, a study of the most general case seems futile, and only cases with small numbers of parameters seem worthy of consideration. For more than two variables, the situation can only become worse. The systems of equations for the undetermined coefficients tends to be large. Algebraic manipulation programs on computer may be the only way to make useful searches other than for a very small number of parameters.

There do exist some little theorems of very special character. Inspections of various examples, including those from Section 5, suggest that certain combinations of terms sometimes appear. For example, we can ask which functions, if any, satisfy a three term relation of the form

$$G - AG[a_1+1] = ??? \qquad \text{... (4.3)}$$

The two terms are chosen because of the similarities of (3.3) and (3.6). An examination shows that if $A = b_2 - a_1$ and $p_1 = 1$, then the expression can be compared with (3.9). If further $q_1 = 2$, $p_3 = q_3 = 0$, the products match. Hence

Theorem 1 — The functions

$$G^{2,1:m_2,n_2;0,0}_{1,2:p_2,q_2;0,0} \qquad \text{... (4.4)}$$

satisfy the less simple contiguous function relation

$$G - (b_2 - a_1)G[a_1+1] + yG[b_1 - 1] = 0. \qquad \text{... (4.5)}$$

Symmetries produce companion results.

We next use (3.5), the analog of (3.6) for e, and (3.12).

Theorem 2 — The functions

$$G^{1,1:0,0;p_3,q_3}_{1,1:0,0;m_3,n_3} \qquad \text{... (4.6)}$$

satisfy the less simple contiguous function relations

$$xG - G[c_1+1] - (b_1 - a_1)\,xG[b_1 - 1] = 0. \qquad \text{... (4.7)}$$

Symmetries again produce companion results.

Theorem 3 — The functions

$$G^{m_1,n_1:0,1;0,1}_{p_1,q_1:1,0;1,0} \qquad \text{... (4.8)}$$

satisfy the less simple contiguous function relation

$$xG[c_1-1] = yG[e_1 - 1]. \qquad \text{... (4.9)}$$

5. Examples

Example 1 — In order to show how one might proceed in an explicit case, we consider the G-function

$$\mathbf{M}\left\{ G^{0,1:0,1;0,1}_{1,1:1,0;1,0}\left[\begin{array}{c|c} x & a:c;e \\ y & b:-;- \end{array}\right]\right\}$$

$$= \Gamma\left[\begin{array}{lll} 1-a-s-t, & 1-c-s, & 1-e-t \\ 1-b-s-t, & & \end{array}\right]. \qquad \text{... (5.1)}$$

First, for the simple relations, the matrix for the function is

$$\left[\begin{array}{cccc} -1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \end{array}\right]. \qquad \text{... (5.2)}$$

As a consequence, a basis for the simple relations is the pair

$$G[a-1] - G[b+1] = (b-a-1)G, \qquad \text{... (5.3)}$$

$$G[a-1] - G[c-1] - G[e-1] = (c+e-a-1)G. \qquad \text{... (5.4)}$$

We can simplify our search for less simple relations, since we can now omit terms which involve $b+1$ or $e-1$. Thus we seek relations of the form

$$(A_0 + A_1x + A_2y)G + B_1G\,[a+1] + B_2G[c+1] + B_3G[e+1]$$

$$+ (C_0 + C_1x + C_2y)\,G[a-1] + (D_0 + D_1x + D_2y)\,G[c-1]$$

$$+ (E_0 + E_1x + E_2y)\,G[b-1] = 0. \qquad \text{... (5.5)}$$

Let

$$\Omega = \Gamma\begin{bmatrix} -a-s-t, & -c-s, & -e-t \\ 1-b-s-t, & & \end{bmatrix}. \qquad \text{... (5.6)}$$

which will become the removable common factor. Next we use the appropriate special cases of (3.3)-(3.13). An examination of terms of degrees 5 and then 4 eliminates five of the coefficients. If we judiciously choose values for s and t, we obtain further relations among the coefficients and ultimately find a set which can be arbitrarily chosen. After extended calculations we thus obtain a basis for the less simple relations of the form (5.5). We list only two, since for each there is a companion relation involving y and e which can be obtained from symmetry.

$$xG - G[c+1] - (b-a)\, xG[b-1] = 0, \qquad \text{... (5.7)}$$

$$G - (b-a)\, G[a+1] - xG[c-1] = 0. \qquad \text{... (5.8)}$$

Example 2 — It is of interest to note that the function

$$G^{0,1:0,1;0,0}_{1,0:1,0;0,0}\begin{bmatrix} a : c; - \\ - : -; - \end{bmatrix} \qquad \text{... (5.9)}$$

satisfies two less simple contiguous relations,

$$yG = G[a+1] = xG[c-1], \qquad \text{... (5.10)}$$

although it satisfies no simple relations.

Example 3 — As an example of how one might proceed in a completely analoguous manner for a series, we consider the function of Volkodavov and Nikolaev[24]. They obtained two recurrence formulas, neither of which were contiguous relations as defined here. Their function,

$$R(\alpha, \beta, \beta', \delta; \gamma, \delta'; x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\alpha)_{n+m}\, (\beta)_n\, (\beta')_m\, (\delta)_n}{(\gamma)_{n+m}\, (\delta')_n} \frac{x^n y^m}{n!\, m!}, \qquad \text{... (5.11)}$$

which is connected with a boundary value problem, is a generalization of the Appell function F_1. The simple contiguous relations are easy to write down by inspection using methods for series analogous to those mentioned in Section 2, see also Buschman[6]. We have

$$\alpha R[\alpha+1] - (\gamma-1)\, R[\gamma-1] = (\alpha-\gamma+1)R, \qquad \text{... (5.12)}$$

$$\beta R\,[\beta+1] - \delta R\,[\delta+1] = (\beta-\delta)\, R, \qquad \text{... (5.13)}$$

$$\beta R\,[\beta+1] - (\delta'-1)\, R\,[\delta'-1] = (\beta-\delta'+1)R, \qquad \text{... (5.14)}$$

$$\alpha R\,[\alpha+1] - \beta R\,[\beta+1] - \beta' R\,[\beta'+1] = (\alpha-\beta-\beta')R. \qquad \text{... (5.15)}$$

A judicious selection of the Ω-function, which will ultimately factor out of the equation is,

$$\Omega = \Gamma\left[\begin{array}{c} \gamma, \delta', 1, 1 \\ \alpha, \beta, \beta', \delta \end{array}\right]$$

$$\Gamma\left[\begin{array}{cccc} \alpha + n + m - 1, & \beta + n - 1, & \beta' + m - 1, & \delta + n - 1 \\ \gamma + n + m - 1, & \delta' + n, & 1 + n, & 1 + m \end{array}\right] \quad \ldots (5.16)$$

By methods analogous to Example 1 and extensive calculations, we obtain three less simple contiguous relations.

$$(1 - y) R - R[\beta' - 1] + (1 - \alpha/\gamma) yR[\gamma + 1] = 0, \quad \ldots (5.17)$$

$$(1 - x)R - \frac{\delta' - \beta}{\delta - \beta} R[\beta - 1] - \frac{\delta' - \delta}{\beta - \delta} R[\delta - 1]$$

$$+ (1 - \alpha/\gamma) yR[\gamma + 1] = 0, \quad \ldots (5.18)$$

$$[(2\alpha - \gamma) - (\delta' - \delta)x] R + (\gamma - \alpha) R[\alpha - 1] - \alpha R[\alpha + 1] + \beta x R[\beta + 1]$$

$$+ \beta' yR[\beta' + 1] + (\gamma' - \gamma)(1 - \delta/\delta') xR[\delta' + 1] = 0. \quad \ldots (5.19)$$

Example 4 — The Horn function

$$G_3(\alpha, \alpha'; x, y) = \sum_{m, n = 0}^{\infty} (\alpha)_{2n-m} (\alpha')_{2m-n} \frac{x^m y^n}{n!\, m!} \quad \ldots (5.20)$$

does not satisfy any simple relations. Because of the factor 2 in the indices, it is convenient to choose

$$\Omega = \Gamma\left[\begin{array}{c} - \\ \alpha, \alpha' \end{array}\right] \Gamma\left[\begin{array}{cc} \alpha + 2n - m - 2, & \alpha' + 2m - n - 2 \\ 1 + m, & 1 + n \end{array}\right]. \quad \ldots (5.21)$$

As in previous examples, this function factors out. An examination of the degrees of the terms in the remaining polynomial identity easily shows that there are no less simple relations of the form

$$(A_0 + A_1 x + A_2 y) G_3 + (B_0 + B_1 x + B_2 y) G_3[\alpha + 1] + (C_0 + C_1 x) G_3[\alpha - 1]$$

$$+ (D_0 + D_1 x + D_2 y) G_3[\alpha' + 1] + (E_0 + E_2 y) G_3[\alpha' - 1] = 0. \quad \ldots (5.22)$$

The method certainly extends to functions of more than two variables, but the computations can become even more cumbersome.

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