Indian J. pure appl. Math., **25**(5); 535-540, May 1994

$\mathcal{A}$-SUMMABILITY AND FRECHET SPACES

MURSALEEN AND Q. A. KHAN

Department of Mathematics, Aligarh Muslim University, Aligarh 202 002

(*Received* 3 *December* 1992; *accepted* 14 *December* 1993)

The idea of $\mathcal{A}$-summability was introduced by H. T. Bell[1] which is a generalization of the usual matrix summability of $A = (a_{nk})$, where $\mathcal{A}$ denotes the sequence of matrices $\{A^i\}$. Let

$$c_{\mathcal{A}} = \{x : \mathcal{A}\, x \text{ is convergent}\},$$

and

$$v_{\mathcal{A}} = \{x : \mathcal{A}\, x \in v\}.$$

In the present paper, we study some topological properties of $c_{\mathcal{A}}$ and $v_{\mathcal{A}}$.

1. INTRODUCTION

Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = \{x_n\}$ respectively, normed by $\| x \| = \sup_n | x_n |$. Let

$$l_1 = \left\{ x : \sum_{n=0}^{\infty} | x_n | < \infty \right\}$$

and

$$v = \left\{ x : \Sigma \; | x_n - x_{n-1} | < \infty, \; (x_{-1} = 0) \right\}.$$

We shall use throughout x to denote a sequence $\{x_n\}$ of complex numbers. In addition, we use the special sequences

$$e_k = \{0, 0, ..., 0, 1, 0, 0, ...\},$$

$$e = \{1, 1, 1, ...\}.$$

Let $A = (a_{nk})$ be any infinite matrix. We consider infinite matrices and sequence-to-sequence transformations of the form :

$$y_n = \sum_{k=0}^{\infty} a_{nk} x_k$$

that is $y = Ax$.

For a sequence space E and a matrix A, we write

$$E_A = \{x : Ax \in E\}.$$

Given two sequence spaces E and F, we say that $A \in (E, F)$ when $F \subset E_A$.

Recently, Bell[1] introduced the idea of $\mathcal{A}$-summability which is a generalization of the usual matrix summability of A.

For $i = 1, 2, \ldots$, let $A^i = (a^i_{nk})$ be an infinite matrix of complex numbers. Let $\mathcal{A}$ denote the sequence of matrices $\{A^i\}$. For a sequence of complex numbers $x = \{x_n\}$ the sequence $y = \{y^i_n\}$ defined by

$$y^i_n = \sum_{k=1}^{\infty} a^i_{nk}\, x_k$$

is called the $\mathcal{A}$-transform of x whenever the series converges for all n and i. A sequence x is said to be $\mathcal{A}$-summable to some number L if $\{y^i_n\}$ converges to L as n tends to ∞ uniformly for $i = 1, 2, \ldots$. L is said to be the $\mathcal{A}$-limit of x, written $\lim_{\mathcal{A}} x = L$ or $\mathcal{A}x \to L$, and we say $\mathcal{A}x$ is convergent to L. Also, that

$$c_{\mathcal{A}} = \{x : \mathcal{A}\, x \text{ is convergent}\}.$$

In Bell[1], necessary and sufficient conditions have been obtained for $\mathcal{A} \in (c, c)$ and $\mathcal{A} \in (l_\infty, c)$. Quite recently, Choudhary and Mursaleen[2] have characterized the matrices $\mathcal{A} \in (v, v)$. For

$$v_{\mathcal{A}} = \{x : \mathcal{A}\, x \in v\}$$

we say that $\{y^{(i)}_n\}$, the family of sequences $\{y^{(i)}_n\}$ with $y^{(i)}_{-1} = 0$ for all i, belongs to v if the series

$$\sum_{n=0}^{\infty} \mid y^{(i)}_n - y^{(i)}_{n-1} \mid \text{ converges uniformly in } i \qquad \ldots (1)$$

and $\lim\limits_{n\to\infty} y^{(i)}_n$ (which under (1) must exist for each i) should take the same value for all i.

The object of this paper is to study some topological properties of $c_{\mathcal{A}}$. We also find a necessary and sufficient condition for $v_{\mathcal{A}}$ to be a conull FK-space, for $\mathcal{A} \in (v, v)$.

2. Frechet Spaces Derived From $c_{\mathcal{A}}$

$\mathcal{A}$-summability is essentially based on a method of assigning a particular number of a double sequence, i.e. let

$$y^i_n = (A^i x)_n.$$

Then "convergence" of $\{y_n^i\}$ that has been discussed is defined as y_n^i "converges" to L if and only if $y_n^i \to L$ as $n \to \infty$, uniformly in i. If we consider $y_n = \{y_n^i : i = 1, 2, ...\}$, i.e. for fixed n, y_n is a sequence. Then this "convergence" can be defined as :

y_n^i "converges" to L if and only if $y_n \to Le$ in $||\cdot||_\infty$. Thus the definition of this method of convergence is closely related to $||\cdot||_\infty$, or the space l_∞ which is a BK-space under $||\cdot||_\infty$.

This relationship with the particular space l_∞ prompts the following definitions.

Definition 2.1 — Suppose X is a topological sequence space. Define

$S(X) = \{Y = \{Y_n\} : Y_n \in X, \text{ for each } n\}$,

$l_\infty(X) = \{Y \in S(X)$: The set $\{Y_n : n = 1, 2, ...\}$ is bounded in the topology of $X\}$,

$C(X) = \{Y \in S(X)$: There exists $t \in X$ such that Y converges to t in the topology of $X\}$,

$\hat{C}(X) = \{Y \in C(X) : Y$ converges to t and $t = Le\}$.

We prove the following :

Theorem 2.1 — Suppose X is a Frechet space with paranorm ρ. Then $S(X)$ is a Frechet space under the topology generated by the family $\{\rho \circ P_n\}$ where $P_n : S(X) \to X$ is given by

$$P_n(Y) = Y_n \text{ for } Y = \{Y_n\}.$$

PROOF : X is a Frechet space. Hence $\prod_{i=1}^{\infty} X_i$ is a Frechet space under the topology generated by $\{\rho \circ P_n\}$. $S(X)$ is defined with $\prod_{i=1}^{\infty} X_i$ in the obvious way.

The following theorem of Garling allows us to show quite easily that all the spaces defined above are Frechet spaces.

Theorem 2.2[3] — Suppose that (E, T) is a Hausdroff locally convex space with topology given by a family P of seminorms. Let Q be a collection of lower semi-continuous extended seminorms on E. Let $F = \{X \in E : q(X) < \infty$ for all $q \in Q\}$. Then if (E, T) is complete, the collection $P \cup Q$ defines a complete locally convex topology on F.

We shall now make the assumption that X is a locally convex FK-space whose topology is given by seminorms $\{p^k : k = 1, 2, ...\}$.

Theorem 2.3 — $l_\infty(X)$ is a Frechet space with topology given by $\{\gamma_k\}$ where $\gamma_k = \sup_n \{p^k \circ P_n(Y)\}$.

PROOF : $l_\infty(X) \subset S(X)$. In fact $l_\infty(X) = \bigcap_{k=1}^{\infty} \{Y \in S(X) : \sup_n p^k \circ P_n(Y) < \infty\}$. Thus by Garling's theorem and a theorem on page 205 in Wilansky[6], $(l_\infty(X), \{p^k \circ P_n\}$

$\bigcup \{\gamma_k\})$ is complete. But $\gamma_k > p^k \circ P_n$ for all n. Thus $(l_\infty(X), \{\gamma_k\})$ is a Frechet space.

Remark : If $(X, \|\cdot\|)$ is a Banach space, then $(l_\infty(X), \|\cdot\|)_\infty$ is a Banach space, where $\|x\|_\infty = \sup_n \|P_n(Y)\|$.

Theorem 2.4 — (i) $C(X)$ is a closed subspace of $l_\infty(X)$.

(ii) $\hat{C}(X)$ is a closed subspace of $C(X)$.

PROOF : (i) Suppose $Y^k \in C(X)$ for each k and $Y^k \to Y$ in $l_\infty(X)$. Let $Y^k = \{y_n^k\}$. Then for each k, $y_n^k \to L^k \in X$. We shall show that $\{L^k\}$ is a Cauchy sequence in X.

Let p^m be any seminorm on X, then

$$p^m(L^k - L^j) \le p^m(L^k - y_n^k) + p^m(y_n^k - y_n^j) + p^m(y_n^j - L^j)$$

$$\le p^m(L^k - y_n^k) + \gamma_m(Y^k - Y^j) + p^m(y_n^j - L^j),$$

$\{Y^k\}$ converges in $l_\infty(X)$, hence $\{Y^k\}$ is Cauchy and $y_n^i \to L^i$ in X for all i. Thus $p^m(L^k - L^j)$ can be made small for each n by choosing k, j large enough so that middle summand is small than for those k, j, choose n large enough that the other two summands are small.

X is complete. Thus there exists $L \in X$ such that $L^k \to L$ in X, as $k \to \infty$. Thus $Y = \{y_n\}$ converges to L in X since for each n,

$$p^m(y_n - L) \le p^m(y_n - y_n^k) + p^m(y_n^k - L^k) + p^m(L^k - L)$$

$$\le \gamma_m(Y - Y^k) + p^m(y_n^k - L^k) + p^m(L^k - L).$$

This can be made arbitrarily small for each m by choosing k so that the first and last summands are small and then choosing n so that the middle summand is small for that particular k.

(ii) The proof is same as in (i) noting that if L^k is a constant sequence for each k, and $L^k \to L$ in X, then L is a constant sequence.

This follows from the fact that X is a K-space, i.e., $L^k \to L$ in X implies $L_j^k \to L_j$ for each j. Thus since $L_j^k = L_i^{k\bullet}$ for all j, i and each k, we have that $L_j = L_i$ for all j, i i.e. L is a constant sequence.

3. $v_{\mathcal{A}}$ AS CONULL *FK*-SPACE

Definition 3.1[5] — Suppose $\mathcal{A} = \{A^i\}$ is a conservative sequence of matrices, define

$$X_{\mathcal{A}} = \lim_{\mathcal{A}} e - \sum_{k=1}^{\infty} \lim_{\mathcal{A}} e_k .$$

If $A^i = A$ for $i = 1, 2, \ldots$, define $X_A = X_{\mathcal{A}} \cdot \mathcal{A}$ (or A) is said to be conull if $X_{\mathcal{A}} = 0$, and coregular if $X_{\mathcal{A}} \neq 0$.

Snyder[4] has observed the following equivalent formulation.

Theorem 3.1[4] — Let $A = (a_{nk})$ be a conservative matrix. For each $\gamma = 1, 2, \ldots$, let

$$\psi^{\gamma} = e - \sum_{k=1}^{\gamma} e_k = (0, 0, \ldots, 0, 1, 1, \ldots).$$

Then A is conull if and only if $\psi^{\gamma} \to 0$ (weakly) in the FK-space c_A.

Definition 3.2[4] — Let X be an FK-space containing e and e_k. X is conull in case

$$\psi^{\gamma} \to 0 \text{ (weakly) in } X,$$

coregular otherwise.

Choudhary and Mursaleen[2] obtained a set of necessary and sufficient conditions for $\mathcal{A} \in (v, v)$. The following theorem characterizes $v_{\mathcal{A}}$ as a conull FK-space.

Theorem 3.2 — Let $\mathcal{A} \in (v, v)$. $v_{\mathcal{A}}$ is a conull FK-space if and only if

$$\lim_{r \to \infty} \sum_{n=0}^{\infty} \left| \sum_{k=r}^{\infty} b_{nk}^{(i)} \right| = 0, \text{ uniformly in } i,$$

where

$$b_{nk}^{(i)} = a_{nk}^{(i)} - a_{n-1,k}^{(i)}, \; n = 1, 2, \ldots$$

and

$$a_{-1,k}^{(i)} = 0, \text{ for all } i, k.$$

PROOF : By Theorem 5 of Wilansky[6] (p.230), every $f \in v_{\mathcal{A}}'$ has the form

$$f(x) = g(\mathcal{A} x) + \sum_{k=1}^{\infty} \alpha_k x_k$$

where $g \in v'$ and $\{\alpha_k\}$ is a sequence with the property that $\sum_{k=1}^{\infty} \alpha_k x_k$ converges for each $x \in v_{\mathcal{A}}$. Since v is congruent to l_1 relative to the matrix $D = (d_{nk})$ where

$$d_{nk} = \begin{cases} 1, & n = k \\ -1, & n = k-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we can write

$$g(y) = t_1 y_1 + \sum_{k=1}^{\infty} t_{k+1} (y_{k+1} - y_k)$$

for some $y \in v$, where $\{t_k\}$ is some bounded sequence. Since $e \in v_{\pi}$, it follows that v_{π} is conull if and only if $\psi^r \to 0$ (weakly) in v, hence if and only if for each i

$$e_i^r \to 0 \quad \text{(weakly) in } l_1,$$

where $e_i^r = (e_{in}^r)$ with

$$e_{in}^r = \sum_{k=r+1}^{\infty} b_{nk}^{(i)}.$$

But weak convergence and strong convergence of a sequence are equivalent in l_1. This completes the proof.

References

1. H. T. Bell, *Proc. Amer. Math. Soc.* **38** (1973), 548-52.
2. B. Choudhary and Mursaleen, *Indian J. pure. appl. Math.* **18** (1987), 823-29.
3. D. J. H. Garling, *Proc. Camp. Phil. Soc.* **63** (1967), 997-1019.
4. A. K. Snyder, *Math. Z.* **90**, (1965), 376-81.
5. A. Wilansky, *Trans. Amer. Math. Soc.* **67** (1949), 59-68.
6. A. Wilansky, *Functional Analysis*, Blaisdell, New York, 1964.