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# THE EFFECT OF MASS AND HEAT TRANSFER ON RAYLEIGH-TAYLOR STABILITY

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The nonlinear Rayleigh-Taylor stability of a liquid layer over a vapour layer of finite depth has been examined using the method of multiple scales and taking into account the effect of heat and mass transfer. These results are more general than the ones obtained earlier by Hsieh[10] who restricted his analysis to potential flows. In the general case it is found that the region of stability increases. The size of the bubbles which detach from the interface will be smaller compared to the ones obtained under restricted flows.

## 1. INTRODUCTION

The stability of superposed fluids has been subject of great attention ever since the poineering work by Lord Rayleigh[1] and by Helmholtz[2] and Kelvin[3] (See : Chandrasekhar[4]). The nonlinear analysis attempted so far are mainly concentrated in the study of two semi-infinite fluids separated by the interface $y = 0$ under ideal conditions, neglecting the dissipative effects (See : Drazin[5], Nayfeh and Saric[6], Weissman[7], Malik and Singh[8]). However, there are various situations where these effects assume significance. For example when the fluid is boiling, whether film, boiling or pool boiling, the motion of the film and the bubbles depends principally on the effect of mass and heat transfer across the interface which plays an important role in determining the flow field and its stability.

Hsieh[9] presented a formulation to deal with interfacial stability problem taking into account mass and heat transfer. In the linear analysis, for Rayleigh-Taylor stability problems of a liquid-vapour system, it is found that the effect of mass and heat transfer tends to reduce the growth rate of instability, even though the criterion for stability is the same as the classical result. Hsieh[10] used the multiple expansion method to study the nonlinear Rayleigh-Taylor stability of a liquid layer over a finite vapour layer. It is observed that the nonlinear effects can indeed increase the range of stability of the system when there is strong heat and mass transfer, while this is not the case for linear Rayleigh-Taylor instability. In the nonlinear analysis, the size of the vapour bubbles which detach from the interface can also be estimated.

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Hsieh[10] studied the character of equilibrium of two incompressible inviscid fluid layers and assumed the flows to be irrotational. It is the purpose of the present paper to examine this problem in the more general case when no restriction is made on the nature of flow i.e. we do not necessarily restrict it to be irrotational.

## 2. Formulation of the Problem

Consider two incompressible, inviscid fluid layers in gravitational field separated by an interface $y = \eta(x, t)$. We shall use superscript 1 and 2 to denote variables in these two fluids, which occupy the region $-h_1 < y < \eta$ and $\eta < y < h_2$ respectively (see Fig. 1). Let the temperatures at $y = -h_1, y = 0$ and $y = h_2$ be $T_1, T_0$ and $T_2$ respectively. The equations governing the system is the $j$th region are :

$$\boldsymbol{u}_t^{(j)} + (\boldsymbol{u}^{(j)} \cdot \nabla)\, \boldsymbol{u}^{(j)} = -\nabla p^{(j)} - g\boldsymbol{e}_y \qquad \dots (1)$$

$$\nabla \cdot \boldsymbol{u}^{(j)} = 0. \qquad \dots (2)$$

where $\boldsymbol{u}^{(j)} = (u^{(j)}, v^{(j)})$ is the velocity, $P^{(j)}$ the fluid pressure, $\rho^{(j)}$ the density and $g$ the acceleration due to gravity, and $p^{(j)} = P^{(j)}/\rho^{(j)}$. At the solid/liquid interface, the normal component of fluid velocities vanishes, i.e.

$$v^{(j)}(x, (-1)^j h_j, t) = 0. \qquad \dots (3)$$

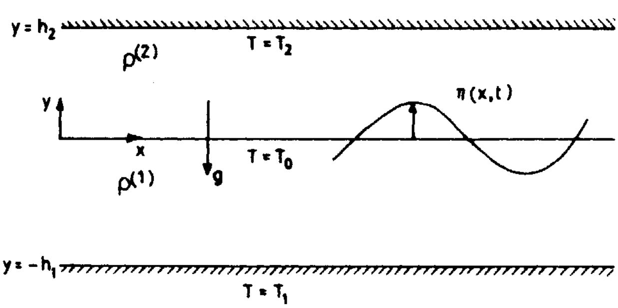


FIG. 1. The schematic configuration of the fluid system.

The interfacial conditions on the interface $y = \eta(x, t)$ are :

$$\rho^{(1)} \left[ \eta_t + u^{(1)} \eta_x - v^{(1)} \right] = \{(2)\} \qquad \dots (4)$$

$$\rho^{(1)} \left[ p^{(1)} - g\eta - \left( 1 + (\eta_x)^2 \right)^{-1} \left( u^{(1)} \eta_x - v^{(1)} \right) \left( \eta_t + u^{(1)} \eta_x - v^{(1)} \right) \right] = \{(2)\} - \sigma \eta_{xx} \left( 1 + (\eta_x)^2 \right)^{-3/2} \qquad \dots (5)$$

$$\rho^{(1)} \left[ \eta_t + u^{(1)} \eta_x - v^{(1)} \right] = -\alpha \left( \eta + \alpha_2 \eta^2 + \alpha_3 \eta^3 \right) \qquad \dots (6)$$

where the notation $\{(2)\}$ on the right-hand sides of these equations is used to denote the same expression as that on the left, except for changing the superscript (1) to (2). The coefficient of surface tension is denoted by $\sigma$. It is to be noted that according to the quasi-equilibrium approximation, the coefficients $\alpha$ and $\alpha_2$ are given by (see Hsieh[9]).

$$\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right) \qquad \text{... (7)}$$

$$\alpha_2 = \left(\frac{1}{h_2} - \frac{1}{h_1}\right) \qquad \text{... (8)}$$

where $G$ is the equilibrium heat flux and $L$ is the latent heat released when the fluid is transformed from phase 1 to phase 2. The coefficients $\alpha, \alpha_2$ and $\alpha_3$ are all of order $O(1)$.

We wish to examine the motions which are small perturbations to the equilibrium state described by $\eta = 0$ and $\partial p_0/\partial y = -g$. We shall expand all the physical quantities in powers of $\varepsilon$, a scaling parameter assumed small, and in order to make these expansions uniform, we introduce slow scales in time, $t_n = \varepsilon^n t$. We then write

$$\Phi(x, y, t; \varepsilon) = \sum_{n=1}^{3} \varepsilon^n \, \Phi_n(x, y, t_0, t_1, t_2) + O(\varepsilon^4) \qquad \text{... (9)}$$

where $\Phi$ is any of the variables $u^{(j)}, v^{(j)}, p^{(j)} - p_0$ or $\eta$. While writing the expansion for $\eta$, it will be noted that $\eta$ depends on $x$ and $t$ and not on $y$. Since eqn. (3) are linear, each $v_n^{(j)}$ satisfies eqn. (3). On substituting these expansions into the remaining equations and equating the coefficients of $\varepsilon^n$, we get equations governing the system in the various orders. They are given for $n = 1$, 2 and 3 in the Appendix.

## 3. Multiscale Expansion Near the Critical Wave Number

We imagine the interface to be perturbed from $y = 0$ to $y = \eta \exp i(kx - \omega t_0)$. In the linearized theory, the system of equations (A.2) to (A.8) admit the solutions:

$$\eta_1 = A_+$$

$$u_1^{(j)} = iT^{(j)} \cosh(k\varphi_j) A_-$$

$$v_1^{(j)} = T^{(j)} \sinh(k\varphi_j) A_+$$

$$p_1^{(j)} = \left(\frac{i\omega}{k}\right) T^{(j)} \cosh(k\varphi_j) A_- \qquad \text{... (10)}$$

where

$$A_\pm = A(t_1, t_2)\, e^{i\theta} \pm \bar{A}\,(t_1, t_2)\, e^{-i\theta} \qquad \text{... (11)}$$

$$T^{(j)} = \frac{(-1)^{j+1}}{\sinh(kh_j)} \left(\frac{\alpha}{\rho^{(j)}} - i\omega\right) \qquad \text{... (12)}$$

and $\theta = kx - \omega t_0$, and $\phi_j = y - (-1)^j h_j$. The frequency $\omega$ and the wave number $k$ satisfy the dispersion relation

$$\omega^2 \left[ \rho^{(1)} \cosh kh_1 \sinh kh_2 + \rho^{(2)} \cosh kh_2 \sinh kh_1 \right]$$
$$+ \; i\alpha\omega \sinh k(h_1 + h_2) + \left[ (\rho^{(2)} - \rho^{(1)})\, gk - \sigma k^3 \right] \sinh kh_1 \sinh kh_2 = 0. \quad \text{... (13)}$$

It is found that criterion for stability is the same as the classical result and the critical wave number is given by

$$k_c = \left[ \frac{g(\rho^{(2)} - \rho^{(1)})}{\sigma} \right]^{1/2}. \quad \text{... (14)}$$

The system is stable for all $k > k_c$. The corresponding critical frequency is zero for this case. To determine an expansion valid near the cut off wave number, let $k = k_c + \varepsilon^2 \mu$, with $\mu = O(1)$. Then eqns. (A.2)-(A.20) remain unchanged except $k$ is replaced by $k_c$ and (A.19) is modified by adding the term $-2\,\sigma\mu\, k_c\, \eta_{1,xx}$. The first order solution will reproduce the linear result for the critical case. The solutions can now be written as

$$\eta_1 = A_+$$
$$u_1^{(j)} = iM^{(j)} \cosh (k\phi_j)\, A_-$$
$$v_1^{(j)} = M^{(j)} \sinh (k\phi_j)\, A_+$$
$$p_1^{(j)} = 0 \quad \text{... (15)}$$

where

$$A_\pm = A\, e^{i\psi} + \bar{A}\; e^{-i\psi} \quad \text{... (16)}$$

$$M_j = \frac{\rho_j\, \alpha}{\sinh kh_j}, \quad \rho_j = \frac{(-1)^{j+1}}{\rho^{(j)}} \quad \text{... (17)}$$

and $\psi = kx$. The second order equations are given in the Appendix. On substituting the first-order solutions given by eqn. (15), the secularity equation for the second order problem is given by

$$\frac{\partial A}{\partial t_1} = 0. \quad \text{... (18)}$$

We obtain for the uniformly valid second-order solutions as :

$$\eta_2 = \frac{\alpha^2}{g\,(\rho^{(2)} - \rho^{(1)})} \sum_{j=1}^{2} \rho_j \left[ (3 + \coth^2 kh_j)\, A\bar{A} + \frac{1}{6} (\coth^2 kh_j - 3)\, A^2 e^{2i\psi} \right] + c.c.$$

$$u_2^{(j)} = ia_v{}^{(j)} \cosh 2k\phi_j + c.\,c.,$$

$$v_2^{(j)} = a_v^{(j)} \sinh 2k\varphi_j + c_v^{(j)} + c.c.,$$

$$p_2^{(j)} = (M^{(j)})^2 \left[ \frac{A^2}{2} e^{2i\psi} - (-1)^{j+1} \cosh(2k\phi_j) A\bar{A} \right] + c.c. \qquad \dots (19)$$

where

$$\left( \rho^{(1)} \frac{\sinh 2kh_1}{\alpha} \right) a_v^{(1)} = \left( \alpha_2 - 2k \coth kh_1 \right) A^2 + \frac{\alpha^2}{6g\,(\rho^{(2)} - \rho^{(1)})} \sum_{j=1}^{2} \left[ \rho_j (3 - \coth^2 kh_j) A^2 \right] \qquad \dots (20)$$

$$\sum_{j=1}^{2} \left[ \rho^{(j)} \sinh(2kh_j)\, a_v^{(j)} + \alpha k \coth(kh_j) A^2 \right] = 0 \qquad \dots (21)$$

$$\frac{\rho^{(1)} c_v^{(1)}}{\alpha} = 2\alpha_2 A\bar{A} + \frac{\alpha^2}{g(\rho^{(2)} - \rho^{(1)})} \sum_{j=1}^{2} \left[ \rho_j (3 + \coth^2 kh_j) A \bar{A} \right] \qquad \dots (22)$$

$$\rho^{(2)} c_v^{(2)} = \rho^{(1)} c_v^{(1)}. \qquad \dots (23)$$

Proceeding to the third-order problem, we obtain after some straightforward reductions the evolution equation for the amplitude :

$$\frac{\alpha}{k} (\coth kh_1 + \coth kh_2) \frac{\partial A}{\partial t_2} + 2\sigma k_c \mu A + \left( \upsilon - \frac{3}{2} \sigma k_c^4 \right) A^2 \bar{A} = 0 \qquad \dots (24)$$

where

$$\upsilon = \upsilon_H + 10\alpha^2 \alpha_2 \left( \frac{1}{\rho^{(1)}} - \frac{1}{\rho^{(2)}} \right) + 2k\alpha^2 \left[ 3 \sum_{j=1}^{2} \{ \rho_j (1 - \coth kh_j) \} - \frac{2}{\rho^{(1)}} \coth kh_1 \right] + \frac{\alpha^4}{6g\,\rho^{(1)} \rho^{(2)}} \sum_{j=1}^{2} \left[ \rho_j (46 + 22 \coth^2 kh_j - 4 \coth kh_j \coth 2kh_j) \right] \qquad \dots (25)$$

$$\upsilon_H = k\alpha^2 \sum_{j=1}^{2} \left[ \rho_j (\coth kh_j \coth 2kh_j - 1)\, (N + \frac{\alpha_2}{k} - 2 \coth kh_j) \right] \qquad \dots (26)$$

$$N = \frac{\alpha^2}{6gk\,(\rho^{(2)} - \rho^{(1)})} \sum_{j=1}^{2} \left[ \rho_j (1 + \coth^2 kh_j) \right] \qquad \dots (27)$$

## 4. Discussion

The expression for $\upsilon$ differs significantly from the one given by Hsieh[10]. This discrepency must be due to the fact that the motions being considered here are quite general and not restricted to potential flows. Since the phase associated with $A$ remains constant, without any loss of generality, we may assume $A$ to be real in eqn. (24). We may then write equation (24) in the form

$$\frac{dA}{dt_2} + (a_1 + a_2 A^2) A = 0 \qquad \text{... (28)}$$

where

$$a_1 = \frac{2\sigma\mu\, k_c\, k}{\alpha\,(\coth kh_1 + \coth kh_2)} \qquad \text{... (29)}$$

$$a_2 = \frac{\left(\upsilon - \frac{3}{2}\sigma\, k_c^4\right) k}{\alpha\,(\coth kh_1 + \coth kh_2)} . \qquad \text{... (30)}$$

From eqn. (28), we conclude that a sufficient condition for stability is $a_2 > 0$, which is due to the effect of finite amplitude. Stability can also be established if $a_1 > 0$ and the initial amplitude is small enough; this is the linear result (See : Hsieh[10]). The stability condition $a_2 > 0$ is equivalent to $\upsilon > (3/2)\,\sigma\, k_c^4$. From the expression of $\upsilon$ it is clear that our $\upsilon$ is larger in comparison with that given by Hsigh[10]. This means that the region of stability increases with the relaxation of the condition of potential flow. Since $\upsilon$ is always positive and $\upsilon$ is larger for the general flows than the potential flows, it is important to note that the radius of the bubbles which will detach from the interface will be smaller as compared to the one which obtain in the case of potential flows. When $\upsilon - (3/2)\,\sigma\, k_c^4 < 0$, then for large times $A_c = \sqrt{a_1/\xi}$, where $\xi = (3/2)\,\sigma\, k_c^4 - \upsilon$. It is found that the wave amplitude first increases and then become saturated. For semi-infinite fluid layers, it is found that $\upsilon$ vanishes. This is quite compatiple with the physical condition that for a semi-infinite layer, the effects of heat and mass transfer will be negligible. Thus the system cannot be stabilized by the finite amplitude effects up to this order. Since the vapour density $\rho^{(1)}$ is usually much smaller than the liquid density $\rho^{(2)}$. We have for $\rho^{(1)} \ll \rho^{(2)}$ :

$$\upsilon = \frac{k\alpha^2}{\rho^{(1)}}\left[\,(N - 2\coth kh_1)(\coth kh_1 \coth 2kh_1 - 1) + 6 - 10\coth kh_1\,\right] \qquad \text{... (31)}$$

where

$$N = \frac{\alpha^2\,(1 + \coth^2 kh_1)}{6\,\rho^{(1)}\,\sigma\, k k_c^3} . \qquad \text{... (32)}$$

If we neglect $\alpha_2$, then the criterion $\upsilon > 0$ is equivalent to

$$N > 2\coth kh_1 + \frac{10\coth kh_1 - 6}{\coth kh_1 \coth 2kh_1 - 1} . \qquad \text{... (33)}$$

*Case I* : $kh_1 << 1$

We have plotted $N$ for various values of $kh_1$ (see Fig. 2). It is found that large values of $N$ are required as compared to the ones obtained by Hsieh[10]. This implies that disturbances of larger wave lengths can be still made stable as compare to the result for $kh_1 << 1$ obtained by Hsieh[10].

*Case II* : $kh_1 >> 1$

According to Hsieh[10] the system can be made stable if $N > 2$. We find that the additional term in equation (33) increases exponentially. This makes it almost impossible to make the system stable for large values of $kh_1$.

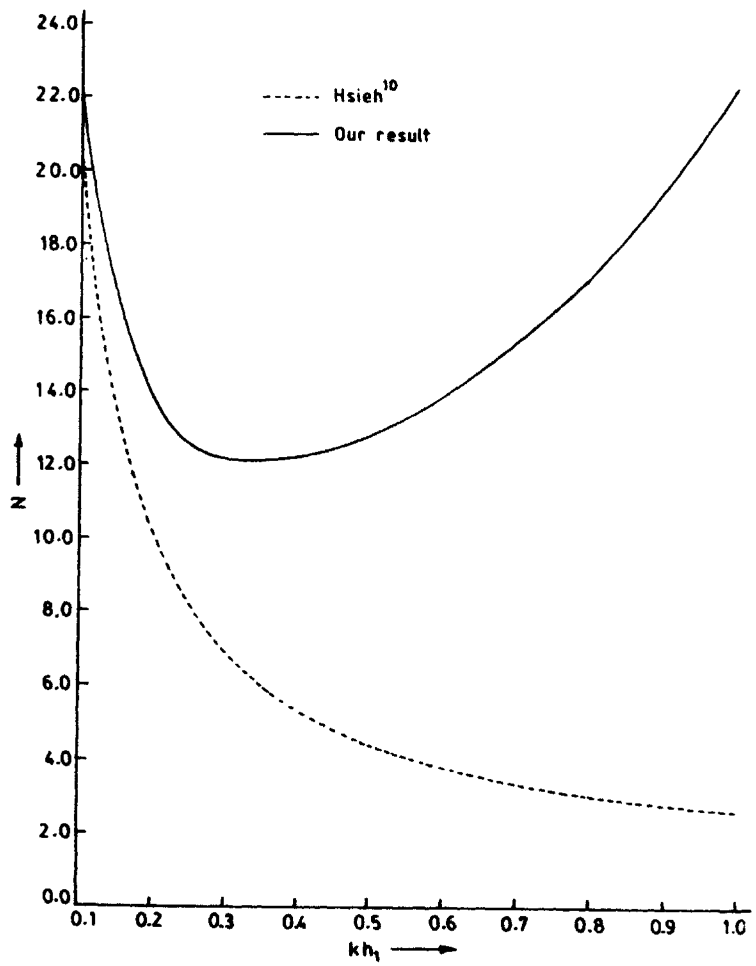


FIG. 2. The behaviour of $N$ as a function of $kh_1$.

Thus we conclude that thinner the vapour layer, easier it is to make the system stable, while with the same heat flux it is almost impossible to stabilize the system for large values of $kh_1$. These results appear to be in confirmity with the physical fact that the effect of heat and mass transfer will clearly be more pronounced for layers of small thickness; this effect tends to become negligible for layers of very large thickness.

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## APPENDIX

We define the following operators :

$$L_i(f, h) = \frac{\partial f}{\partial t_i} + \frac{\partial h}{\partial x}$$

$$M_i(f, h) = \frac{\partial f}{\partial t_i} + \frac{\partial h}{\partial y}$$

$$N(f, h) = \frac{\partial f}{\partial x} + \frac{\partial h}{\partial y}$$

$$\nabla_i = u_i^{(j)} \frac{\partial}{\partial x} + v_i^{(j)} \frac{\partial}{\partial y}$$

$$D_\mu f_i = \frac{\partial f_i}{\partial \mu_1} + \frac{\partial f_{i-1}}{\partial \mu_2} + \dots + \frac{\partial f_1}{\partial \mu_i},$$

$$\nabla_\mu (f_i, h_i) = \frac{\partial f_i}{\partial \mu_0} + \frac{\partial f_{i-1}}{\partial \mu_1} + \dots + \frac{\partial f_1}{\partial \mu_{i-1}} - h_i,$$

$$P(f, h) = f - gh,$$

$$R(f_i, h_i) = u_1^{(1)} \frac{\partial f_i}{\partial x} + u_2^{(1)} \frac{\partial f_{i-1}}{\partial x} - \eta_1 \frac{\partial h_i}{\partial y} - \eta_2 \frac{\partial h_{i-1}}{\partial y},$$

$$T(f, h) = \eta_1 \frac{\partial \eta_1}{\partial x} \frac{\partial f}{\partial x} - \eta_2 \frac{\partial h}{\partial y} - \frac{\eta_1^2}{2} \frac{\partial^2 h}{\partial y^2},$$

$$W(f, h) = u_1^{(1)} \frac{\partial f}{\partial x} - \eta_1 \frac{\partial h}{\partial y}. \qquad \text{... (A. 1)}$$

First-order equations :

$$L_0 (u_1^{(j)}, p_1^{(j)}) = 0, \qquad \text{... (A. 2)}$$

$$M_0 (v_1^{(j)}, p_1^{(j)}) = 0, \qquad \text{... (A. 3)}$$

$$N (u_1^{(j)}, v_1^{(j)}) = 0, \qquad \text{... (A. 4)}$$

$$v_1^{(1)} (x, -h_1, t) = 0, \; v_1^{(2)} (x, h_2, t) = 0 \qquad \text{... (A. 5)}$$

boundary conditions at $y = 0$ :

$$\rho^{(1)} \Delta_t (\eta_1, v_1^{(1)}) = \{(2)\}, \qquad \text{... (A. 6)}$$

$$\rho^{(1)} P (p_1^{(1)}, \eta_1) = \{(2)\} - \sigma \eta_{1,xx}, \qquad \text{... (A. 7)}$$

$$\rho^{(1)} \Delta_t (\eta_1, v_1^{(1)}) = -\alpha \eta_1. \qquad \text{... (A. 8)}$$

Second-order equations :

$$L_0 (u_2^{(j)}, p_2^{(j)}) = -D_t u_1^{(j)} - \nabla_1 u_1^{(j)} \qquad \text{... (A. 9)}$$

$$M_0 (v_2^{(j)}, p_2^{(j)}) = -D_t v_1^{(j)} - \nabla_1 v_1^{(j)} \qquad \text{... (A. 10)}$$

$$N (u_2^{(j)}, v_2^{(j)}) = 0 \qquad \text{... (A. 11)}$$

$$v_2^{(1)} (x, -h_1, t) = 0, \; v_2^{(2)} (x, h_2, t) = 0. \qquad \text{... (A. 12)}$$

boundary conditions at $y = 0$ :

$$\rho^{(1)} \left[ \Delta_t (\eta_2, v_2^{(1)}) + R(\eta_1, v_1^{(1)}) \right] = \{(2)\} \qquad \text{... (A. 13)}$$

$$\rho^{(1)} \left[ P (p_2^{(1)} \eta_2) + v_1^{(1)} \Delta_t (\eta_1, v_1^{(1)}) \right] = \{(2)\} - \sigma \eta_{2,xx} \qquad \text{... (A. 14)}$$

$$\rho^{(1)} \left[ \Delta_t (\eta_2, v_2^{(1)}) + R(\eta_1, v_1^{(1)}) \right] = -a(\eta_2 + \alpha_2 \eta_1^2). \qquad \text{... (A. 15)}$$

Third-order equations

$$L_0 (u_3^{(j)}, p_3^{(j)}) = -D_t u_2^{(j)} - \nabla_1 u_2^{(j)} - \nabla_2 u_1^{(j)} \qquad \text{... (A. 16)}$$

$$M_0 (v_3^{(j)}, p_3^{(j)}) = -D_t v_2^{(j)} - \nabla_1 v_2^{(j)} - \nabla_2 v_1^{(j)} \qquad \text{... (A. 17)}$$

$$N(u_3^{(j)}, v_3^{(j)}) = 0, \qquad \text{... (A. 18)}$$

$$v_3^{(1)}(x, -h_1, t) = 0, \quad v_3^{(2)}(x, h_2, t) = 0, \qquad \text{... (A. 19)}$$

boundary conditions at $y = 0$ :

$$\rho^{(1)}\left[\Delta_t(\eta_3, v_3^{(1)}) + R(\eta_2, v_2^{(1)}) + T(u_1^{(1)}, v_1^{(1)})\right] = \{(2)\}, \qquad \text{... (A. 20)}$$

$$\rho^{(1)}\Big[P(p_3^{(1)}, \eta_3) + v_1^{(1)}\Delta_t(\eta_2, v_2^{(1)}) - 2v_1^{(1)} v_2^{(1)} -$$

$$v_2^{(1)}\Delta_t(\eta_1, v_1^{(1)}) + (3v_1^{(1)} + \nabla_t(\eta_1, v_1^{(1)}))\, W(\eta_1, v_1^{(1)})\Big]$$

$$= \{(2)\} - \sigma\left[\eta_{3,xx} - \frac{3}{2}\eta_{1,xx}(\eta_{1,x})^2\right], \qquad \text{... (A. 21)}$$

$$\rho^{(1)}\left[\Delta_t(\eta_3, v_3^{(1)}) + R(\eta_2, v_2^{(1)}) + T(u_1^{(1)}, v_1^{(1)})\right]$$

$$= -\alpha(\eta_3 + 2\alpha_2\eta_1\eta_2 + \alpha_3\eta_1^3). \qquad \text{... (A. 22)}$$