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UNSTEADY FLOW OF A NON-NEWTONIAN FLUID DOWN AN OPEN INCLINED CHANNEL

P. R. SHARMA AND HARISH KUMAR

Department of Mathematics, University of Rajasthan, Jaipur 302 004

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The unsteady flow of a Walters fluid (Model B′) down an open inclined channel under gravity has been investigated analytically. The exact solution of velocity distribution has been obtained by using Laplace transform and Finite Fourier Sine Transform techniques. A uniform tangential stress is applied at the free surface in the direction of flow. The velocity distribution has been obtained taking different forms of time dependent pressure gradient $g(t)$ viz. (i) constant, (ii) exponentially decreasing function of time, and (iii) Cosine function of time. It has been observed that the unsteady flow will never occur in case of a Newtonian fluid when the pressure gradient is taken to be constant, but the opposite behaviour is observed in the non-Newtonian fluid case. The fluid flow ultimately becomes steady under the exponential decreasing form of the pressure gradient in both non-Newtonian and Newtonian fluids.

1. INTRODUCTION

The flow of a liquid in an open inclined channel with a free surface has a wide application in the designs of drainage, irrigation canals, flood discharge channels and coating to paper rolls etc. Hence the flow of a liquid in an open inclined channel with a free surface under gravity has long been studied experimentally and several interesting empirical results have been reported by many investigator[1,3-6,10]. The steady laminar flow of a viscous fluid flowing down an open inclined channel has been discussed by Satya Prakash[7]. Gupta *et al.*[2] have studied the flow of a viscous fluid through a porous medium down an open inclined channel. Unsteady laminar flow of an incompressible viscous fluid between porous, parallel flat plates has been investigated by Singh[8], taking (i) both plates are at rest and (ii) Generalised plane Couette flow.

Non-Newtonian fluids have wide importance in the present day technology and industries. The Walters fluid is one of such fluid. The constitutive equations governing motion of Walters fluid (Model B′) are :

$$P_{ik} = -p\, g_{ik} + P_{ik}' \qquad \text{... (1.1)}$$

$$P_{ik}' = 2\eta_0\, e^{ik} - 2\, K_0\, e'^{ik}. \qquad \text{... (1.2)}$$

The equations of motion and continuity are

$$\rho\left(\frac{\partial v_i}{\partial t}+v^j v^i_{,j}\right)=-P_{,j}+P^{ij}_{,j} \qquad \text{... (1.3)}$$

and

$$v^i_{,i}=0 \qquad \text{... (1.4)}$$

where P_{ik} is the stress tensor, ρ the density, P the pressure, g_{ik} the metric tensor of a fixed coordinate system x^i, v^i the velocity vector, e'^{ik} in the contravariant form is

$$e'^{ik}=\frac{\partial e^{ik}}{\partial t}+v^j e^{ik}_{,j}-v^k_{,j}e^{ij}-v^i_{,j}e^{jk}. \qquad \text{... (1.5)}$$

It is the convected derivative of the deformation rate tensor e^{ik} defined by

$$2e_{ik}=v_{i,k}+v_{k,i}. \qquad \text{... (1.6)}$$

Here, η_0 is the limiting viscosity at small rates of shear which is given by

$$\eta_0=\int_0^\infty N(\tau)\,d\tau \qquad \text{... (1.7)}$$

and

$$K_0=\int_0^\infty \tau N(\tau)\,d\tau. \qquad \text{... (1.8)}$$

$N(\tau)$ being the relaxation spectrum as introduced by Walters[11, 12]. This idealised model is a valid approximation of Walter's fluid (Model B′) taking very short memory into account so that terms involving

$$\int_0^\infty \tau^n N(\tau)\,d\tau,\quad n\geq 2 \qquad \text{... (1.9)}$$

have been neglected.

In the present analysis, the unsteady flow of a Walters fluid (Model B′) down an open inclined channel under gravity has been investigated. A uniform tangential stress is applied at the free surface in the direction of flow. The exact solution of velocity distribution has been obtained by using Laplace transform and Finite Fourier Sine transform techniques. Here it is assumed that (i) the fluid flows in the steady state for $t \leq 0$, (ii) unsteady state occurs at $t > 0$, and (iii) the unsteady motion is influenced by time dependent pressure gradient. The initial velocity is taken non-zero and equal to that of steady state. The velocity distribution has been obtained in some particular cases i.e. when (i) $g(t) = C$, (ii) $g(t) = C\,e^{-bt}$, and (iii) $g(t) = C\cos bt$, where b and C are constants. It has been observed that the unsteady state never occurs in case of a Newtonian fluid when the pressure gradient is taken to be constant but the opposite behaviour is observed in case of the non-Newtonian fluid. The fluid flow ultimately becomes steady under the exponential decreasing form of the pressure gradient in both non-Newtonian and Newtonian fluids.

2. Governing Equations of Motion

Consider the flow of a Walter's fluid (Model B') down an open inclined channel of width $2a$ and depth d under gravity. A uniform tangential stress S is applied at the free surface. The bottom of the channel is inclined at an angle $\beta (0 \leq \beta \leq \pi/2)$ with the horizontal. The x-axis is taken along a central line in the free surface and pointing downwards lengthwise, and y-axis along a line pointing downwards and perpendicular to the free surface, and z-axis along a line in the free surface perpendicular to the x-axis. Let u be the velocity component in the direction of the x-axis and the other velocity components be zero.

The equations of continuiuty and motion for the unsteady, viscous, incompressible, Walters fluid (Model B') flowing down an open inclined channel at $t > 0$ are :

$$\frac{\partial u}{\partial x} = 0 \quad \text{... (2.1)}$$

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \rho g \sin \beta + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) - K_0 \left(\frac{\partial^3 u}{\partial t \, \partial y^2} + \frac{\partial^3 u}{\partial t \, \partial z^2} \right) \quad \text{... (2.2)}$$

$$0 = -\frac{\partial p}{\partial y} + \rho g \cos \beta \quad \text{... (2.3)}$$

$$0 = -\frac{\partial p}{\partial z} \quad \text{... (2.4)}$$

where ρ is the density, g the acceleration due to gravity, p is the pressure, μ the coefficient of viscosity and K_0 the non-Newtonian parameter.

The boundary conditions are :

$$
\begin{aligned}
&t \leq 0; \quad && u = u_0 \\
&t > 0; \quad && z = \pm a, \; u = 0 \\
&y = 0, \quad && \mu \frac{\partial u}{\partial y} = S \\
&y = d, \quad && u = 0
\end{aligned}
\quad \text{... (2.5)}
$$

where u_0 is the initial velocity.

Introducing the following non-dimensional quantities :

$$u' = \frac{u}{U}, \; x' = \frac{x}{d}, \; y' = \frac{y}{d}, \; z' = \frac{z}{d}, \; t' = \frac{t}{d^2},$$

$$p' = \frac{p}{\rho U^2}, \; K = \frac{K_0}{\rho d^2}, \; S' = \frac{S}{U^2}$$

in the equation (2.2) and after dropping the dashes, we have

$$\frac{\partial u}{\partial t} = -R\frac{\partial p}{\partial x} + \frac{R}{F}\sin\beta + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - K\left[\frac{\partial^3 u}{\partial t\,\partial y^2} + \frac{\partial^3 u}{\partial t\,\partial z^2}\right] \qquad \text{... (2.6)}$$

where U is the characteristic velocity, K $(= K_0/\rho\, d^2)$ is the non-dimensional non-Newtonian parameter, $R(= Ud/\nu)$ is the Reynolds number and $F(= U^2/gd)$ is the Froud number.

The boundary conditions are reduced to :

$$\begin{aligned} t \le 0\,; &\quad u = u_0 \\ t > 0\,; &\quad z = \pm\, l(= a/d),\ u = 0 \\ y = 0, &\quad \partial u/\partial y = SR \\ y = 1, &\quad u = 0 \end{aligned} \qquad \text{... (2.7)}$$

3. Method of Solution

Assuming $-R\dfrac{\partial p}{\partial x} + \dfrac{R}{F}\sin\beta = g(t)$ at $t > 0$

$$= P \quad \text{at} \quad t \le 0 \qquad \text{... (3.1)}$$

Substituting $z = \dfrac{2l\xi}{\pi} - l$ in eqn. (2.6), gives,

$$\frac{\partial u}{\partial t} = g(t) + \frac{\partial^2 u}{\partial y^2} + \frac{\pi^2}{4l^2}\frac{\partial^2 u}{\partial \xi^2} - K\left(\frac{\partial^3 u}{\partial t\,\partial y^2} + \frac{\pi^2}{4l^2}\frac{\partial^3 u}{\partial t\,\partial \xi^2}\right) \qquad \text{... (3.2)}$$

and the boundary conditions are reduced to

$$\begin{aligned} t \le 0\,; &\quad u = u_0 \\ t > 0; &\quad \xi = 0, \pi,\ u = 0 \\ y = 0, &\quad \partial u/\partial y = SR \\ y = 1, &\quad u = 0. \end{aligned} \qquad \text{... (3.3)}$$

Now, since u_0 is the initial velocity i.e. at $t \le 0$, therefore, taking $g(t) = p$ in eqn. (3.2), u_0 is given by

$$u_0 = \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{4l^2 p(1-\cos n\pi)}{\pi^2 n^3}\left(1 - \frac{\cosh Qy}{\cosh Q}\right) - \frac{S\,R(1-\cos n\pi)}{nQ}\,\frac{\sinh Q\,(1-y)}{\cosh Q}\right]\sin n\xi \qquad \text{... (3.4)}$$

where $Q = n\pi/2l$.

Now, to solve eqn. (3.2), we take Laplace transform of equation (3.2) with respect to t (Sneddon[9]) defined as

$$\bar{u}(y, \xi, s) = \int_0^\infty u(y, \xi, t)\, e^{-st}\, dt\,; \quad s > 0. \qquad \text{... (3.5)}$$

We get

$$\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\pi^2}{4l^2}\frac{\partial^2 \bar{u}}{\partial \xi^2} - \frac{s}{1-ks}\bar{u} = \frac{1}{(1-ks)}\left(KP - \tilde{g}(s) - u_0\right) \qquad \text{... (3.6)}$$

where $\tilde{g}(s) = \int_0^\infty g(t)\, e^{-st}\, dt.$

On taking the finite Fourier sine transform of eqn. (3.6) with respect to ξ (Sneddon[9]) defined as

$$\bar{u}^*(y, M, s) = \int_0^\pi \bar{u}(y, \xi, s) \sin M\xi \, d\xi. \qquad \text{... (3.7)}$$

We get

$$\frac{d^2\bar{u}^*}{dy^2} - H^2\bar{u}^* = \frac{(1-\cos M\pi)}{M(1-KS)}\left[KP - \tilde{g}(s) - \frac{P}{Q^2}\right]$$
$$+ \frac{P(1-\cos M\pi)}{M(1-KS)\,Q^2}\,\frac{\cosh Qy}{\cosh Q} + \frac{SR(1-\cos M\pi)}{(1-Ks)\,QM}\,\frac{\sinh Q(1-y)}{\cosh Q} \qquad \text{... (3.8)}$$

where $H^2 = \dfrac{s}{1-Ks} + Q^2.$

Now, the boundary conditions are reduced to

$$y = 0\,, \ \frac{d\,\bar{u}^*}{dy} = \frac{SR(1-\cos M\pi)}{s\,M}$$

$$y = 1, \ \bar{u}^* = 0. \qquad \text{... (3.9)}$$

Integrating eqn. (3.8) under the boundary conditions (3.9), we get

$$\bar{u}^* = \frac{(1-\cos M\pi)}{M}\left[\frac{P\cosh Hy}{sH^2\cosh H} - \frac{P}{sQ^2}\,\frac{\cosh Qy}{\cosh Q} + \frac{P}{Q^2H^2(1-Ks)}\right.$$
$$\left. + \frac{(g(s)-KP)}{H^2(1-Ks)}\left(1 - \frac{\cosh Hy}{\cosh H}\right) - \frac{SR}{SQ}\,\frac{\sinh Q(1-y)}{\cosh Q}\right] \qquad \text{... (3.10)}$$

Now, inverting the finite Fourier sine transform as given by Sneddon[9],

$$\bar{u}(y, \xi, s) = \frac{2}{\pi} \sum_{M=1}^{\infty} \bar{u}^{*}(y, M, s) \sin M\xi$$

in eqn. (3.10), we get

$$\bar{u}(y, \xi, s) = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \left[\frac{P}{s H^2} \frac{\cosh Hy}{\cosh H} - \frac{P}{sQ^2} \frac{\cosh Qy}{\cosh Q} + \frac{P}{Q^2 H^2 (1-Ks)} + \frac{(g(s)-KP)}{H^2(1-Ks)} \left(1 - \frac{\cosh Hy}{\cosh H} \right) - \frac{SR}{sQ} \frac{\sinh Q(1-y)}{\cosh Q} \right] \sin M\xi. \quad \text{... (3.11)}$$

On inverting the Laplace Transform as defined by Sneddon[9],

$$u(y, \xi, t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \bar{u}(y, \xi, s)\, e^{st}\, dt$$

in eqn. (3.11), we obtain

$$u = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \sin M\xi \left\{ \sum_{r=0}^{\infty} \frac{4P(-1)^r}{\pi(2r+1)} \times \frac{\cos a_r y}{(a_r^2 + Q^2)} + \int_0^t h(u)\, g(t-u)\, du - \frac{SR}{Q} \frac{\sinh Q(1-y)}{\cosh Q} \right\} \quad \text{... (3.12)}$$

where

$$h(u) = \sum_{r=0}^{\infty} \frac{(-1)^r 4 \cos a_r y\, e^{-A_r u}}{(2r+1)\pi \left\{ 1 - K(a_r^2 + Q^2 \right\}},$$

$$a_r = (2r+1)\frac{\pi}{2},$$

and $$A_r = \frac{(a_r^2 + Q^2)}{1 - K(a_r^2 + Q^2)}.$$

4. Particular Cases

When $g(t) = C$

Case I : Using in eqn. (3.12), we get

$$u = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \sin M\xi \left[\sum_{r=0}^{\infty} \left(P \frac{4(-1)^r \cos a_r y}{\pi(2r+1)(Q^2+a_r^2)} e^{-A_r t} - C \right) \right.$$

$$\left. + \frac{C}{Q^2}\left(1 - \frac{\cosh Qy}{\cosh Q}\right) \frac{SR \sinh Q(1-y)}{Q \cosh Q} \right] \quad ...(4.1)$$

(a) Letting $K \to 0$, $g(t) = P = C$, the velocity is given by

$$u = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \sin M\xi \left[\frac{C}{Q^2}\left(1 - \frac{\cosh Qy}{\cosh Q}\right) - \frac{SR}{Q} \frac{\sinh Q(1-y)}{\cosh Q} \right] \quad ... (4.2)$$

which is equivalent to u_0 and in good agreement with Satya Prakash[7].

Case II : When $g(t) = C\, e^{-bt}$, $b > 0$ Using in eqn. (3.12), we get

$$u = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \sin M\xi \left\{ \sum_{r=0}^{\infty} \frac{4P(-1)^r}{\pi(2r+1)} \frac{\cos a_r y}{(a_r^2+Q^2)} e^{-A_r t} \right.$$

$$+ \sum_{r=0}^{\infty} \frac{4C(-1)^r \cos a_r y}{\pi(2r+1)[1-K(a_r^2+Q^2)]} \frac{(e^{-bt} - e^{-A_r t})}{(A_r - b)}$$

$$\left. - \frac{SR}{Q} \frac{\sinh Q(1-y)}{\cosh Q} \right\} \quad ... (4.3)$$

Case III : When $g(t) = C \cos bt$

Using eqn. (3.12), we have

$$u = \frac{2}{\pi} \sum_{M=1}^{\infty} \frac{(1-\cos M\pi)}{M} \sin M\xi \left[\sum_{r=0}^{\infty} \frac{4P(-1)^r}{\pi(2r+1)} \frac{\cos a_r y}{(a_r^2+Q^2)} e^{-A_r t} \right.$$

$$+ \sum_{r=0}^{\infty} \frac{4C(-1)^r \cos a_r y}{\pi(2r+1)[1-K(a_r^2+Q^2)]} \times \frac{A_r}{(b^2 A_r^2+1)}$$

$$\left. \times (bA_r \sin bt + \cos bt - e^{-A_r t}) - \frac{SR}{Q} \frac{\sinh Q(1-y)}{\cosh Q} \right] \quad ... (4.4)$$

5. Numerical Discussions

From eqn. (4.1), it is observed that the non-Newtonian fluid particles are more faster in comparision to Newtonian fluid particles. Equation (4.2) shows that the unsteady flow will never occur in the Newtonian fluid when the pressure gradient is taken to be constant and which is equal to the initial velocity u_0, but the opposite behaviour is seen in the non-Newtonian fluid case. It is observed that the fluid flow ultimately becomes steady under the exponentially decreasing form of the pressure gradient in both Newtonian and non-Newtonian fluids. Further, if we take $S = 0$ i.e. $\frac{du}{dy} = 0$ at $y = 0$, both the fluids ultimately come to rest under the exponentially decreasing form of the pressure gradient.

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