Indian J. pure appl. Math., 25(5); 559-566, May 1994

EXISTENCE OF PERIODIC ORBITS OF COLLISION IN THE PHOTOGRAVITATIONAL CIRCULAR RESTRICTED PROBLEM OF THREE BODIES IN THREE-DIMENSIONAL CO-ORDINATE SYSTEM

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(*Received* 16 *April* 1993; *accepted* 7 *September* 1993)

This paper deals with the motion of an infinitesimal mass in the gravitational field of the sun and planet taking into account the effect of radiation pressure of the sun on the planet as well as the infinitesimal mass. The existence of periodic orbits of collision in the photogravitational circular restricted problem of three bodies in three-dimensional co-ordinate system has been studied here. We shall assume that the third co-ordinate q_3 is of the order of μ.

1. INTRODUCTION

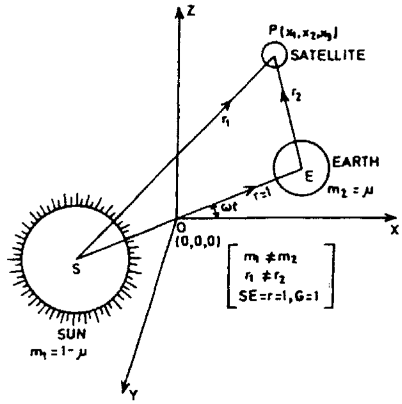
Das and Das[3] have studied the problem of the existence of periodic orbits of collision in the photogravitational circular restricted problem of three bodies in three-dimensional co-ordinate system taking into account the effect of radiation pressure on both the infinitesimal mass as well as the planet. The authors have taken the mean motion to be unity (which we shall see, is not so) and the mass reduction factors of solar radiation due to the sun to be the same for the infinitesimal mass and the planet (which is not so). Due to the difference in the size (and hence the cross-sectional areas) of the infinitesimal mass and the planet and also the difference in the sun-infinitesimal mass and sun-planet distances, we shall take the mass reduction factors β_1 and β_2 to be different, see Fig. 1.

Also Das and Das[3] have taken the same generating function used by Bhatnagar[1] without proper modifications. In the light of above short commings we have restudied the problem in three-dimensional co-ordinate system.

Let us take the sum of the masses of the two primaries to be unity. Since the sun is the source of radiation, hence if μ is the mass parameter of the planet; then the mass parameter of the sun is $\beta_2(1-\mu)$. We shall also take the distance between the primaries to be unity and the centre of mass of the system as the origin. Then the co-ordinates of the sun and planet are $\left(\frac{\mu}{\mu+\beta_2(1-\mu)}, 0, 0\right)$ and $\left(-\frac{\beta_2(1-\mu)}{\mu+\beta_2(1-\mu)}, 0, 0\right)$ respectively. The effect of radiation pressure on the infinitesimal mass is also taken here. The mass parameter (in this case) is $\beta_1(1-\mu)$. When the sun is the source of radiation, proceeding as in classical case, the mean motion ω can be written as

$$\omega^2 = \frac{G(m_1\beta_2 + m_2)}{r^3}$$

In dimensionless variables, $m_1 = 1-\mu$, $m_2 = \mu$; $r = 1, G = 1$, therefore $\omega^2 = \mu + \beta_2(1-\mu)$.



The figure shows the different effect of the solar radiation pressure on both the infinitesimal mass (Satellite) as well as the planet.

2. Equations of Motion

The canonical equations of motion in the photogravitational circular restricted problem of three bodies in three-dimensional synodic co-ordinates (x_1, x_2, x_3) are given by

$$\frac{dx_i}{dt} = \frac{\partial\Omega}{\partial p_i} \qquad (i = 1, 2, 3)$$

$$\frac{dp_i}{dt} = -\frac{\partial\Omega}{\partial x_i} \qquad \text{... (1)}$$

where the corresponding Hamiltonian Ω is given by

$$\Omega = \frac{1}{2}(p_1^2 + p_2^2 + p_3^2) + \omega(p_1 x_2 - p_2 x_1) - \frac{\beta_1(1-\mu)}{r_1} - \frac{\mu}{r_2} = C \qquad \dots (2)$$

$$r_1^2 = \left(x_1 - \frac{\mu}{\omega^2}\right)^2 + x_2^2 + x_3^2 \qquad \dots (3)$$

$$r_2^2 = \left(x_1 - \frac{\beta_2(\mu-1)}{\omega^2}\right)^2 + x_2^2 + x_3^2 \qquad \dots (4)$$

$$p_1 = \dot{x}_1 - \omega x_2 \qquad \dots (5)$$

$$p_2 = \dot{x}_2 + \omega x_1 \qquad \left(\cdot = \frac{d}{dt} \right)$$

$$p_3 = \dot{x}_3 \; .$$

For regularization at $r_1 = 0$ Das and Das[3] used the same expression of the generating function S as used by Bhatnagar[1] due to which r_1 and r_2 become the functions of μ and β_2. To get rid of μ and β_2, we therefore modify it accordingly and we take the generating function as

$$S = (\mu/\omega^2 + q_1^2 - q_2^2)\, p_1 + 2 q_1 q_2 p_2 + q_3 p_3 \qquad \dots (6)$$

such that $x_i = \dfrac{\partial S}{\partial p_i}$

$$(i = 1, 2, 3)$$

$$Q_i = \frac{\partial S}{\partial q_i} \qquad \dots (7)$$

where Q_i $(i = 1, 2, 3)$ are the moment *a* associated with the new generalized co-ordinates q_i $(i = 1, 2, 3)$.

Let us introduce a new independent variable τ, defined by

$$dt = r_1 \, d\tau \, , \; t = 0 \; \text{ at } \tau = 0. \qquad \dots (8)$$

With the help of above equations, the regularised equations of motion are

$$\frac{dq_i}{d\tau} = \frac{\partial K}{\partial Q_i}$$

$$(i = 1, 2, 3)$$

$$\frac{dQ_i}{d\tau} = -\frac{\partial K}{\partial q_i} \qquad \dots (9)$$

where the new Hamiltonian K is given by

$$K = r_1(\Omega - C)$$

$$= \frac{1}{8}(Q_1^2 + Q_2^2 + Q_3^2) + \frac{1}{2} r_1 (Q_1 q_2 - Q_2 q_1 - 2C_0) - 1$$

$$+ \; e_1 - \frac{1}{4} r_1 e_2 (Q_1 q_2 - Q_2 q_1)$$

$$+ \; \mu \left[\frac{r_1 e_2}{4} (Q_1 q_2 - Q_2 q_1) - \frac{1}{2} (Q_1 q_2 + Q_2 q_1) \right.$$

$$\left. + \frac{1}{4} e_2 (Q_1 q_2 + Q_2 q_1) \; - \; 1 \; + \; e_1 - \frac{r_1}{r_2} - r_1 C_1 \right] \qquad \ldots (10)$$

$\beta_1 = 1 - e_1, \; \beta_2 = 1 - e_2$ (Choudhary[2])

$| \, e_1 \, | \ll 1. \qquad | \, e_2 \, | \ll 1.$

$C = C_0 + \mu C_1$

The quantity K given by eqn. (10) is zero along any solutions of the differential equation (9) and vice-versa.

Every solution of these equations must belong to the $K = 0$ manifold. In other words, the first integral K = constant obtained from eqns (9) has a value which is not arbitrary. It must be zero.

We now propose to write the Hamiltonian, K in the form

$K = K_0 + K_1 + \mu K_2$, where

$$K_0 = \frac{1}{8} (Q_1^2 + Q_2^2 + Q_3^2) + \frac{1}{2} r_1 (Q_1 q_2 - Q_2 q_1 - 2C_0) \; - \; 1 \; = \; - \; \varepsilon \; < \; 0$$

$$K_1 = e_1 - \frac{1}{4} r_1 (Q_1 q_2 - Q_2 q_1) e_2$$

$$K_2 = e_1 \frac{e_2}{4} r_1 (Q_1 q_2 - Q_2 q_1) - \frac{1}{4} (2 - e_2) (Q_1 q_2 + Q_2 q_1) - \frac{r_1}{r_2} - r_1 C_1 \; - \; 1.$$

Now K_0 has the same expression as in Bhatnagar[1]. We shall assume that K_0 is negative i.e, ($K_0 = -\varepsilon < 0 \;\; | \, \varepsilon \, | < 1$). The solutions corresponding to K_0 are given by Bhatnagar[1]. In terms of canonical elements (l, L, g, G, h, H) defined by Bhatnagar[1] the Hamiltonian K_1 is written as

$$K_1 = e_1 + \frac{1}{2} e_2 \, a \, G \, (1 - e \cos l) \qquad \ldots (11)$$

Hamiltonian equations of motion for the Hamiltonian K_1 are

$$\frac{d\alpha_r}{d\tau} = \frac{\partial K_1}{\partial \delta_r} \qquad (r \; = \; 1, \; 2, \; 3)$$

$$\frac{d\delta_r}{d\tau} = - \frac{\partial K_1}{\partial \alpha_r} \qquad \ldots (12)$$

where $\alpha_1 = l, \; \alpha_2 = g, \; \alpha_3 = h, \; \delta_1 = L, \; \delta_2 = G$ and $\delta_3 = H$.

From eqns. (11) and (12)

$$\frac{dl}{d\tau} = \frac{\partial K_1}{\partial L} = 0 \qquad \dots (13)$$

$$\frac{dg}{d\tau} = \frac{\partial K_1}{\partial G} = \frac{1}{2} a e_2 (1 - e \cos l) \qquad \dots (14)$$

$$\frac{dh}{d\tau} = \frac{\partial K_1}{\partial H} = 0 \qquad \dots (15)$$

$$\frac{dL}{d\tau} = -\frac{\partial K_1}{\partial l} = -\frac{1}{2} a\, e e_2 G \sin l \qquad \dots (16)$$

$$\frac{dG}{d\tau} = -\frac{\partial K_1}{\partial g} = 0 \qquad \dots (17)$$

$$\frac{dH}{d\tau} = -\frac{\partial K_1}{\partial h} = 0. \qquad \dots (18)$$

Solutions corresponding to K_0 gives $q_1, q_2, q_3, Q_1, Q_2, Q_3$ in terms of the canonical elements l, g, h, L, G, H. Equations (14) and (16) define g, L as functions of time, which on substitution in the solution corresponding to K_0 gives us the values of $q_1, q_2, q_3, Q_1, Q_2, Q_3$ as functions of canonical elements and time. This will give us the solutions corresponding to the Hamiltonian $(K_0 + K_1)$.

Now the canonical equation of motion for the Hamiltonian K are

$$\frac{dl}{d\tau} = \frac{\partial K}{\partial L} = [H^2 - 2(G + C_0)]^{1/2} + \mu \frac{\partial K_2}{\partial L}$$

$$\frac{dg}{d\tau} = \frac{\partial K}{\partial G} = -\frac{L}{[H^2 - 2(G + C_0)]^{1/2}} + \frac{1}{2} a e_2 (1 - e \cos l) + \mu \frac{\partial K_2}{\partial G}$$

$$\frac{dh}{d\tau} = \frac{\partial K}{\partial H} = \frac{LH}{[H^2 - 2(G + C_0)]^{1/2}} + \mu \frac{\partial K_2}{\partial H}$$

$$\frac{dL}{d\tau} = \frac{\partial K}{\partial l} = -\frac{1}{2} a\, e e_2 G \sin l - \mu \frac{\partial K_2}{\partial l} \qquad \dots (19)$$

$$\frac{dG}{d\tau} = -\frac{\partial K}{\partial g} = -\mu \frac{\partial K_2}{\partial g}$$

$$\frac{dH}{d\tau} = -\frac{\partial K}{\partial h} = -\mu \frac{\partial K_2}{\partial h}$$

where $$K_2 = -1 + e_1 - a(1 - e \cos l)\left[\frac{1}{2} G e_2 + \frac{1}{r_2} + C_1\right]$$

$$-\frac{1}{2}(2-e_2)\,[eL\sin l\sin 2\theta + G\cos 2\theta] \qquad \text{... (20)}$$

$$r_2^2 = 1 + 2a(1-e\cos l)\cos 2\theta + a^2(1-e\cos l)^2 + O(\mu^2) \qquad \text{... (21)}$$

and $$q_3 = h - \frac{H(L^2-G^2)^{1/2}}{[H^2-2(G+C_0)]}\sin l. \qquad \text{... (22)}$$

Equation (19) form the basis of general theory of perturbation for the problem in question.

3. Existence of Periodic Orbits When $\mu \neq 0$

The orbits of the infinitesimal mass will be periodic for $\mu \neq 0$ if the following conditions (Duboshin, 1964) are satisfied :

$$\frac{\partial[R_1]}{\partial\omega_i} = 0 \qquad \text{... (23)}$$

$$\frac{\partial[R_1]}{\partial a_i} = 0 \qquad \text{... (24)}$$

and $$\frac{\partial(q_2, q_3, \eta_1, \eta_2, \eta_3)}{\partial(\lambda_2\cdot\lambda_3\cdot\gamma_1\cdot\gamma_2\cdot\gamma_3)} \neq 0 \text{ when } \mu = \gamma_i = \lambda_i = 0 \qquad \text{... (25)}$$

$$(i = 1, 2, 3).$$

The general solution in the neighbourhood of the generating solutions are given by

$$\begin{aligned} x_i &= a_i + \gamma_i + q_i(\rho) \\ y_i &= \eta_i^{(0)} + \omega_i + \lambda_i + \eta_1^{(p)} \end{aligned} \quad (i = 1, 2, 3) \qquad \text{... (26)}$$

where $x_1 = L,\ x_2 = G,\ x_3 = H,\ y_1 = l,\ y_2 = g,\ y_3 = h.$

The left-hand side of inequality (25) can be written as

$$D = \begin{vmatrix} \frac{\partial^2[R_1]}{\partial\omega_2^2} & \frac{\partial^2[R_1]}{\partial\omega_3\,\partial\omega_2} \\ \frac{\partial^2[R_1]}{\partial\omega_2\,\partial\omega_3} & \frac{\partial^2[R_1]}{\partial\omega_3^2} \end{vmatrix} \begin{vmatrix} \frac{\partial^2 R_0}{\partial a_1^2} & \frac{\partial^2 R_0}{\partial a_2\,\partial a_1} & \frac{\partial^2 R_0}{\partial a_3\,\partial a_1} \\ \frac{\partial^2 R_0}{\partial a_1\,\partial a_2} & \frac{\partial^2 R_0}{\partial a_2^2} & \frac{\partial^2 R_0}{\partial a_3\,\partial a_2} \\ \frac{\partial^2 R_0}{\partial a_1\,\partial a_3} & \frac{\partial^2 R_0}{\partial a_2\,\partial a_3} & \frac{\partial^2 R_0}{\partial a_3^2} \end{vmatrix} = D_1 \times D_2.$$

Taking zero order terms, we have

$$r_1 = a,\ r_2^2 = 1 + a^2 + 2\,a\cos 2\theta$$

$$R_0 = L[H^2 - 2\,(G + C_0)]^{1/2} - 1 + e_1 + 1/a\, l_2\, G/2$$

$$= a_1\,[a_3^2 - 2(a_2 + C_0)]^{1/2} - 1 + e_1 + \frac{e_2\, a_1\, a_2}{2[-2(a_2 + C_0)]^{1/2}} \quad \ldots (27)$$

$$[R_1] = -1 + e_1 - \frac{1}{2}\, e_2\, a\, G - \left(1 - \frac{1}{2}\, e_2\right) G\cos 2\theta - \frac{a}{r_2} - aC_1 \quad \ldots (28)$$

Where $a = \dfrac{L}{[-2\,(a_2 + C_0)]^{1/2}}$.

It can easily be shown that

$$D_2 = -\frac{[2\,a_1\, a_3^2 + a_1\, a_4 - a_3^2]}{\lambda^2}\left[\frac{1}{\sqrt{\lambda}} + \frac{2e_2\,(a_2 + 2C_0)}{a_4\, 3/2}\right]$$

$$-\frac{e_2^2\,(a_2 + 2C_0)^2\,(a_1\, a_3^2 + a_1\, a_4 - a_3^2)}{\lambda^{3/2}\, a_4^3} + \frac{e_1\, a_1\, a_3^2\,(a_2 + 4C_0)}{2\lambda\, a_4^{5/2}} \neq 0 \quad \ldots (29)$$

where $\lambda = [a_3^2 - 2(a_2 + C_0)]$ & $a_4 = [-2(a_2 + C_0)]$.

From (28)

$$\frac{\partial[R_1]}{\partial\omega_1} = \frac{\partial[R_1]}{\partial\omega_2} = \left[\left(G - \frac{a^2}{r_2^3}\right) - \frac{1}{2}\, e_2\, G\right]\sin 2\theta$$

as $2\theta = \omega_1 + \omega_2 + \eta_1^{(0)}\,\rho + \eta_2^{(0)}\,\rho,\quad [q_1 = \rho\cos\theta\,,\ q_2 = \rho\sin\theta]$.

Now $\dfrac{\partial[R_1]}{\partial\omega_1} = \dfrac{\partial[R_1]}{\partial\omega_2} = 0$ gives either $r_2^3 = \dfrac{2ar_1}{G(2 - e_2)}$ or $2\theta = 0, \pi$.

Again $\dfrac{\partial^2[R_1]}{\partial\omega_2^2} = \left[\dfrac{G(2 - e_2)}{2} - \dfrac{a^2}{r_2^3}\right]\cos 2\theta - \dfrac{3a^3\sin^2 2\theta}{r_2^5}$.

Then either $2\theta = 0,\ \pi$ or $r_2^3 = \dfrac{2ar_1}{G(2 - e_2)}$ shows that $\dfrac{\partial^2\,[R_1]}{\partial\omega_2^2} \neq 0$.

Also $\dfrac{\partial\,[R_1]}{\partial\omega_3} = \dfrac{r_1}{r_2^2} \times \dfrac{\partial r_2}{\partial\omega_3} - \left(\dfrac{1}{2}\, e_2\, G + \dfrac{1}{r_2} + C_1\right)\dfrac{\partial r_1}{\partial\omega_3}$

$$= \left(\frac{r_1}{r_2^3} - \frac{Ge_2}{2r_1} - \frac{1}{r_1\, r_2} - \frac{C_1}{r_1}\right) q_3\ [\text{as } r_1^2 = a^2 + q_3^2]$$

$$= Bq_3$$

where $$B = \frac{r_1}{r_2^3} - \frac{Ge_2}{2r_1} - \frac{1}{r_1 r_2} - \frac{C_1}{r_1}.$$

Therefore, $\frac{\partial [R_1]}{\partial \omega_3} = 0$, gives, either $B = 0$ or $q_3 = 0$.

Now $\frac{\partial^2 [R_1]}{\partial \omega_3^2} = B \frac{\partial q_3}{\partial \omega_3} + q_3 \frac{\partial B}{\partial \omega_3} \neq 0$ when either $B = 0$ or $q_3 = 0$

Also $\frac{\partial^2 [R_1]}{\partial \omega_2 \partial \omega_3} = \frac{\partial^2 [R_1]}{\partial \omega_3 \partial \omega_2} = \frac{3ar_1 \sin 2\theta \, q_3}{r_2^5} \neq 0$ when either $2\theta = 0, \pi$

or $$r_2^3 = \frac{2ar_1}{G(2-e_2)}.$$

Therefore, $D_1 \neq 0$. i.e. $D = D_1 \times D_2 \neq 0$.

Thus the conditions (23) and (25) are satisfied.

Since $[R_1]$ is independent of a_1, a_2, a_3 and hence $\frac{\partial [R_1]}{\partial a_i} = 0$.

Hence all the conditions for the existence of periodic orbits are satisfied when $\mu \neq 0$.

Proceeding exactly as in Bhatnagar[1], it can easily be shown that Levi-Civita's[6] condition for collision when $\mu \neq 0$ is also satisfied.

ACKNOWLEDGEMENT

The authors are grateful to Dr R. K. Choudhary for his valuable suggestion during the preperation of this paper.

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