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THERMAL STRESSES IN A THICK ELASTIC SLAB CONTAINING AN ANNULAR CRACK

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We consider the problem of determining symmetrical stress distribution in a thick elastic slab containing an annular crack opened by heat load. It is assumed that the deformation is symmetrical. The mixed boundary value problem is reduced to the solution of triple integral equations, which are then reduced to the solution of infinite set of linear simultaneous equations. This set is solved numerically for unknown coefficients and first ten roots are considered. The quantities of physical interest are expressed in terms of these known coefficients and some graphs are also plotted. Mori *et al.* have attempted the same problem but their thermal conditions are stated in a different way. The method of solution is also different and involving as compared to present work.

1. INTRODUCTION

Moss and Kobayashi[1] were first to study the stress distribution near the tip of an annular crack using approximate method. Shibuya *et al.*[2, 3] devised a novel technique for determining stress distribution near the tips of annular crack. Kumar and Atsumi[4, 5], using the technique of Shibuya *et al.*, studied stress distribution in the vicinity of an annular crack situated symmetrically in an infinite transversely isotropic medium and transversely isotropic slab.

In many engineering problems, it is essential to know the interaction of mechanical defects with thermal change, see Kassir and Sih[6]. This book also provides detailed account of thermal problems. Not many attempts have been made to study the effect of thermal load of annular crack only because of the complexity involved in the analysis. However, Kumar and Hiremath[7, 8] were first to study stress distribution when heated annular punch is indented on semi-infinite elastic solid and axisymmetric indentation of thick elastic plate by a pair of heated annular punches. Recently, Kumar and Padmshri[9] have studied stress distribution in the vicinity of an annular crack, contained in an isotropic infinite elastic medium, and opened by constant heat load. Mori *et al.*[10] have studied the distribution of thermal stresses in a slab containing annular crack in the middle plane. The surfaces of the slab are subjected to a prescribed temperature distribution which is odd in Z and crack

surfaces are perfectly insulated. The following two cases are considered :

$$\text{(i)}\ \theta(r,h) = \begin{cases} \theta_0, & 0 \le r < a \\ 0, & r > a \end{cases}$$

$$\frac{\partial}{\partial z}\theta(r,0) = 0, \quad m < r < 1$$

$$\theta(r,0) = 0, \quad 0 \le r < m, \quad r > 1$$

$$\text{(ii)}\ \theta(r,h) = 0, \quad 0 \le r < \infty$$

$$\frac{\partial}{\partial z}\theta(r,0) = -T_0, \quad m < r < 1$$

$$\theta(r,0) = 0, \quad 0 \le r < m, \quad r > 1$$

where θ_0 and T_0 are constants and $\theta(r,z)$ is temperature function.

The triple integral equations for heat conduction and thermoelastic problems are reduced to singular integral equations of first kind. The resulting equations are reduced to two infinite system of simultaneous equations in terms of the series of Chebyshev polynomials of first kind and their weight functions. The method appears to be involving.

However in the present paper, using the technique developed in Kumar and Padmshri[9], the authors have determined the axisymmetric stress distribution in the vicinity of annular crack opened by heat load and contained in isotropic thick slab. The problem is formulated in section 2 and results widely used are given in section 3. The temperature field is determined in next section whereas thermoelastic problem is solved in section 5. The section 6 deals with quantities of physical interest, particularly stress intensity factors K_1 and K_2. The last section gives method of numerically solving infinite set of simultaneous equations. Some graphs are also plotted.

2. Formulation of the Problem

We shall consider the temperature and displacement fields in a perfectly thick slab which is conducting heat. The slab is assumed to be isotropic and homogeneous as far as thermal and mechanical properties are concerned. It is assumed that there

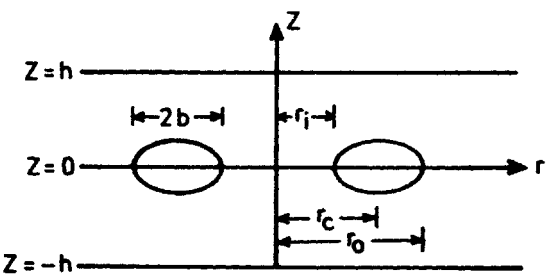

Fig. 1. Geometry of crack in slab.

is symmetry about z axis and free boundaries of slab are $z = \pm h$ (see Fig. 1). The annular crack is supposed to occupy the region $r_i \leq r \leq r_0, z = 0$ of the slab, where r_i and r_0 are respectively the inner and outer radii of the annular crack. For symmetrical deformation of the slab, the displacement vector is assumed to have components (u, o, w) and non-vanishing components of stress may be denoted by $\sigma_{rr}, \sigma_{zz}, \sigma_{\theta\theta}, \sigma_{rz}$. Because of symmetry, it is only sufficient to consider solution in the region $0 \leq z \leq h$. The boundary conditions may be written as

$$\sigma_{zz}(r, 0) = 0, \; r_i \leq r \leq r_0 \qquad \ldots (2.1)$$

$$W(r, 0) = 0, \; r < r_i, \; r > r_0 \qquad \ldots (2.2)$$

$$\sigma_{rz}(r, 0) = 0, \; 0 \leq r < \infty. \qquad \ldots (2.3)$$

The boundary of slab is supposed to be stress free so that :

$$\sigma_{rz}(r, h) = \sigma_{zz}(r, h) = 0. \qquad \ldots (2.4)$$

In addition, we have on $z = h$

$$T(r, h) = 0 \qquad \ldots (2.5)$$

were $T(r, z)$ denotes the temperature field in the slab.

We shall consider following two types of temperature conditions.

Case (a) : The temperature on crack surface is prescribed

$$T(r, 0) = T_0, \; r_i \leq r \leq r_0 \qquad \ldots (2.6)$$

$$\partial T/\partial z)_{z=0} = 0, \; r < r_i, \; r > r_0 \qquad \ldots (2.7)$$

where T_0 is constant temperature.

Case (b) : The flux of heat is prescribed, we have

$$\partial T/\partial z)_{z=0} = -T_1, \quad r_i \leq r \leq r_0 \qquad \ldots (2.8)$$

$$\partial T/\partial z)_{z=0} = 0, \quad r \leq r_i, \; r > r_0. \qquad \ldots (2.9)$$

Basic Equations of Thermoelasticity

The basic equations of equilibrium for an elastic solid conducting heat may be written from[9]

$$2(1-\eta)\,[\partial^2 u/\partial r^2 + r^{-1}\,\partial u/\partial r - r^{-2}\,u] + (1-2\eta)\,\partial^2 u/\partial z^2 + \partial^2 \omega/\partial r\,\partial z = 2\alpha_t(1+\eta)\,\partial T/\partial r \qquad \ldots (2.10)$$

$$(1-2\eta)\,[\partial^2 w/\partial r^2 + r^{-1}\,\partial w/\partial r] + 2(1-\eta)\,\partial^2 w/\partial z^2 + \partial/\partial z\,(\partial u/\partial r + r^{-1}\,u) = 2\alpha_t(1+\eta)\,\partial T/\partial z \qquad \ldots (2.11)$$

where α_t is coefficient of linear expansion of slab and η is its Poisson's ratio. The temperature field $T(r, z)$ in the absence of heat sources and in steady state is governed by

$$\partial^2 T/\partial r^2 + r^2\, \partial T/\partial r + \partial^2 T/\partial z^2 = 0. \qquad \text{... (2.12)}$$

The required components of stress are

$$\sigma_{rz} = \mu(\partial u/\partial z + \partial w/\partial r) \qquad \text{... (2.13)}$$

$$\sigma_{zz} = [2\mu/(1-2\eta)]\,[(1-\eta)\,\partial w/\partial z + \eta(\partial u/\partial r + r^{-1}u) - (1+\eta)\alpha_t T] \qquad \text{... (2.14)}$$

where $\mu = E/2(1+\eta)$ and E is Young's modulus of slab.

The solutions of above equations appropriate to slab problem are

$$T(r, z) = 1/\alpha_t\,(1+\eta) \int_0^\infty (Ae^{sz} + Be^{-sz})\, s\, J_0(sr)\, ds \qquad \text{... (2.15)}$$

$$w(r, z) = \int_0^\infty [\ \{s(C + Dz) - (3-4\eta)D + 2A\}\ e^{sz} - \{s(E + Fz) + (3-4\eta)F + 2B\}\ e^{-sz}\ ]\, J_0(sr)\, ds \qquad \text{... (2.16)}$$

$$\sigma_{zz}(r, z) = 2\mu \int_0^\infty [\ \{s(C + Dz) - 2(1-\eta)D + A\}\ e^{sz} + \{s(E + Fz) + 2(1-\eta)F + B\}\ s^{-sz}\ ]\, s\, J_0(sr)\, ds \qquad \text{... (2.17)}$$

$$\sigma_{rz}(r, z) = 2\mu \int_0^\infty [\ \{s(C + Dz) - (1-2\eta)D + A\}\ e^{sz} - \{s(E + Fz) + (1-2\eta)F + B\}\ e^{-sz}\ ]\, s\, J_0'(sr)\, ds. \qquad \text{... (2.18)}$$

3. Important Results

The integral formula for Bessel functions [Gradshteyn and Ryzhik[11], pp. 695]

$$\int_0^\infty sJ_0(sr)\, Z_n(s)\, ds = \begin{cases} 0, & r < r_i,\ r > r_0 \\ \cos n\phi/(\pi\, br_c \sin\phi), & r_i \le r \le r_0 \end{cases} \qquad \text{... (3.1)}$$

gives us

$$\int_0^\infty G_n(s) J_0(sr)\, ds = \begin{cases} 0, & r < r_i,\ r > r_0 \\ 2 \sin n\phi/(\pi b r_c), & r_i \le r \le r_0 \end{cases} \quad \ldots (3.2)$$

where

$$Z_n(s) = J_n(sr_c) J_n(sb) \quad \ldots (3.3)$$

$$G_n(s) = s\,[Z_{n-1}(s) - Z_{n+1}(s)] \quad \ldots (3.4)$$

$$r^2 = r_c^2 + b^2 - 2br_c \cos\phi, \quad 2r_c = r_i + r_0 \quad \ldots (3.5)$$

$2b = r_0 - r_i$ and $0 \le \phi \le \pi$.

It is simple matter to derive

$$G_n(s) = (2n/br_c) \frac{\partial}{\partial s} Z_n(s)$$

and

$$\int_0^\infty s\, G_n(s) J_0(sr)\, ds = (-2n/br_c)\,[I_0^n + r\,\partial/\partial r\,(I_0^n)] \quad \ldots (3.6)$$

where

$$I_m^n = \int_0^\infty J_m(sr) Z_n(s)\, ds. \quad \ldots (3.7)$$

We shall frequently use following integral formula (Gradshteyn and Ryzhik[11], pp. 667)

$$\int_0^\infty J_\nu(as) J_\mu(xs) J_\alpha(ys)\, ds$$

$$= \frac{\Gamma\left(\frac{1+\sigma+\nu}{2}\right)}{a\,\Gamma(\mu+1)\,\Gamma(\alpha+1)\,\Gamma((1-\sigma+\nu)/2)} (x/a)^\mu x$$

$$(y/a)^\alpha\, 2F_1((1+\sigma+\nu)/2,\ (1+\sigma+\nu)/2,\ \mu+1;\ \sin^2\phi)$$

$$2F_1\left((1+\sigma-\nu)/2,\ \left(\frac{1+\sigma+\nu}{2}\right),\ \alpha+1;\ \sin^2\psi\right) \quad \ldots (3.8)$$

where $2F_1$ is Gaussian hypergeometric function defined by :

$$2F_1(\beta, \eta, r; z) = \Gamma(r)/(\Gamma(\beta)\,\Gamma(\eta)) \sum_{m=0}^{\infty} \Gamma(\beta+m)\,\Gamma(\eta+m)\, z^m/\Gamma(r+m)\, m!$$

and

$$\sigma = \mu + \alpha, \quad x = a \sin\phi \cos\psi, \quad y = a \cos\phi \sin\psi$$

$$a > 0, \ 0 < \phi, \ \psi < \pi/2, \ \mathrm{Re}(\alpha + \mu + \nu) > -1$$

$$\begin{bmatrix} \psi \\ \phi \end{bmatrix} = (1/2)\,[\sin^{-1}(x+y)/a \pm \sin^{-1}(y-x)/a]. \qquad \ldots (3.9)$$

In subsequent sections $2F_1$, is denoted by F only. Using (3.5), we can write

$$J_0(sr) = Z_0(s) + 2 \sum_{m=1}^{\infty} Z_m(s) \cos m\phi\,, \quad (r_i \le r \le r_0) \qquad \ldots (3.10)$$

4. Determination of Temperature Field

We shall determine the temperature field for two type of heat conditions described in section 2.

Case (*A*) : The boundary conditions (2.5) is satisfied if

$$A = -B\, e^{-2sh} \qquad \ldots (4.1)$$

and boundary conditions (2.6) and (2.7) give rise to following triple integral equations for determination of unknown $B(s)$

$$\int_0^{\infty} \tanh sh\, K(s)\, J_0(sr)\, ds = (1+\eta)\,\alpha_t\, T_0, \quad r_i \le r \le r_0 \qquad \ldots (4.2)$$

$$\int_0^{\infty} s\, K(s)\, J_0(sr)\, ds = 0, \ r < r_i, \ r > r_0 \qquad \ldots (4.3)$$

where

$$K(s) = s\, B(s)\, (1 + e^{-2sh}). \qquad \ldots (4.4)$$

To solve the triple integral equations, we assume that the temperature function $T(r, 0)$ can be expressed by cosine Fourier series in the region $r_i \le r \le r_0$ (or $0 \le \phi \le \pi$). In view of relations (3.1) and (3.5), we find that (4.3) is automatically satisfied if $K(s)$ is chosen as

$$K(s) = \sum_{n=1}^{\infty} b_n'\, Z_n(s) \qquad \ldots (4.5)$$

and (4.2) reduces to

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} \tanh sh\, Z_n(s)\, J_0(sr)\, ds = 1 \qquad \ldots (4.6)$$

where

$$b_n = b_n' / (1 + \eta)\, \alpha_t\, T_0. \qquad \ldots (4.7)$$

The use of (3.10) in (4.6), gives us :

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} \tanh sh\, Z_n(s)\, Z_m(s)\, ds = \delta_{0m} \qquad (m = 0, 1, 2, \ldots) \qquad \ldots (4.8)$$

where δ_{0m} is kronecker delta. Equation (4.8) represents an infinite set of linear simultaneous equation if unknown coefficients b_n. This set is solved numerically by the method outlined in section 7.

This determines $K(s)$ and in turn $B(s)$. We can now express $T(r, 0)$ in the region $r < r_i$ and $r > r_0$ as shown below :

$$T(r, 0) = T_0 \sum_{n=1}^{\infty} b_n \left[\int_0^{\infty} Z_n(s)\, J_0(sr)\, ds - \int_0^{\infty} \{2e^{-2sh}/(1 + e^{-2sh})\} Z_n(s)\, J_0(sr)\, ds \right]. \qquad \ldots (4.9)$$

It is worth mentioning here that the first term of (4.9) coincides with the expression of $T(r, 0)$ derived in (9) and the second term shows effect of free boundary. Using (3.7) in eqn. (4.9), we get

$$T(r, 0) = T_0 \sum_{n=1}^{\infty} b_n [I_0^n - I(r)] \qquad \ldots (4.10)$$

where

$$I(r) = \int_0^{\infty} [2e^{-2sh}/(1 + e^{-2sh})]\, Z_n(s)\, J_0(sr)\, ds. \qquad \ldots (4.11)$$

The value of $I(r)$ can be found using Gauss-Laguerre integration formula. The value of I_0^n for $r < r_i$ and $r > r_0$ are given by eqns. (4.7) and (4.8) of Kumar and Padmshri[9] or eqns. (6.6) and (6.8) of the present paper.

Case (B) : The boundary conditions (2.8) and (2.9) are satisfied if

$$\int_0^{\infty} sQ(s)\, J_0(sr)\, ds = \begin{cases} T_1\, \alpha_t\, (1 + \eta), & r_i \le r \le r_0 \\ 0, & r < r_i,\ r > r_0 \end{cases} \qquad \ldots (4.12)$$

where

$$Q(s) = (1 + e^{-2sh})\, sB(s). \qquad \ldots (4.13)$$

The Hankel inversion gives us

$$Q(s) = T_1 \alpha_t (1 + \eta) \int_{r_i}^{r_0} r J_0 (sr)\, ds. \qquad \ldots (4.14)$$

Now, we can express $T(r, 0)$ as

$$T(r, 0) = T_1 \int_0^\infty [r_0 J_1 (sr_0) - r_i J_1 (sr_i)] \times \{(1 - e^{-2sh})/(1 + e^{-2sh})\}\ J_0 (sr)\, ds. \qquad \ldots (4.15)$$

5. THERMOELASTIC PROBLEM

The boundary condition (2.4) is satisfied if

$$s(C + hD) - (1 - 2\eta)D + A = \{s(E + hF) + (1 - 2\eta)F + B\}\ e^{-2sh} \qquad \ldots (5.1)$$

$$s(C + hD) - 2(1 - \eta)D + A = -\{s(E + hF) + 2(1 - \eta)F + B\}\ e^{-2sh}. \qquad \ldots (5.2)$$

The boundary condition (2.3) is satisfied if

$$sC - (1 - 2\eta)D + A = sE + (1 - 2\eta)F + B \qquad \ldots (5.3)$$

where A is given by (4.1). The function $B(s)$ has already been determined in section 4. Thus, we are left with only four functions $C(s)$, $D(s)$, $F(s)$ and $E(s)$. Any three can be determined from (5.1)-(5.3).

The boundary conditions (2.1) and (2.2) are satisfied if

$$\int_0^\infty [1 + M(sh)]\, N(sh)\, s J_0 (sr)\, ds = p(r), \quad r_i \le r \le r_0 \qquad \ldots (5.4)$$

$$\int_0^\infty N(sh) J_0 (sr)\, ds = 0, \quad r < r_i, \quad r > r_0 \qquad \ldots (5.5)$$

where

$$P(r) = \int_0^\infty sL(sh)\, B J_0 (sr)\, ds \qquad \ldots (5.6)$$

$$L(sh) = \frac{(5 - 4\eta)\, e^{-2sh} - (3 - 4\eta)}{p_0 + p_1 e^{-2sh} + p_2 e^{-4sh}} \qquad \ldots (5.7)$$

$$M(sh) = \frac{p_3 + p_4 e^{-2sh} + p_5 e^{-4sh}}{p_0 + p_1 e^{-2sh} + p_2 e^{-4sh}} \qquad \ldots (5.8)$$

$$N(sh) = \frac{p_0 + p_1 e^{-2sh} + p_2 e^{-4sh}}{1-(1-2sh)\, e^{-2sh}} F - \{(3-4\eta)-(5-4\eta)\, e^{-2sh}\} B \quad \ldots (5.9)$$

$$p_0 = 2(1-\eta)\,\{(3-4\eta)-4(1-2\eta)\, sh\}$$

$$p_1 = 8(1-\eta)\,\{(1-2\eta)+sh+s^2 h^2\}$$

$$p_2 = -2(1-\eta)\,\{(3-4\eta)-4(1-2\eta)\, sh-4\, s^2 h^2\}$$

$$p_3 = -5+14\eta-8\eta^2+4(1-2\eta)^2\, sh$$

$$p_4 = -2(5-12\eta+8\eta^2)+4(1-2\eta)^2\, sh$$

$$p_5 = 7-14\eta+8\eta^2-4(1-2\eta)\,(3-2\eta)\, sh-8(2-\eta)\, s^2 h^2.$$

Equations (5.4) and (5.5) represent a set of triple integral equations for determination of $N(sh)$. For the solution of this set we shall follow the procedure given in Kumar and Padmshri[9]. Thus, we utilize the fact that $w(r, 0)$ is continuous on the crack surface $r_i \le r \le r_0$ (or $0 \le \phi \le \pi$) and is zero outisde ($r < r_i$ and $r > r_0$) and the change of variable defined by (3.5) to express $w(r, 0)$ on the crack surface by a Fourier sine series with respect to ϕ as follows :

$$w(r, 0) = (2/\pi b r_c) \sum_{n=1}^{\infty} a_n \sin n\phi, \quad 0 \le \phi \le \pi. \quad \ldots (5.10)$$

Equations (3.2) and (5.10) show that (5.5) is satisfied automatically if

$$N(sh) = \sum_{n=1}^{\infty} a_n\, G_n(s) \quad \ldots (5.11)$$

and (5.4) can be expressed as

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} s\, G_n(s)\,[1+M(sh)]\, J_0(sr)\, ds = p(r). \quad \ldots (5.12)$$

Using the result (3.9) for $J_0(sr)$ and comparing coefficients of cos $m\phi$ on two sides, we get

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} s\, G_n(s)\,[1+M(sh)]\, J_m(sr_0)\, J_m(sb)\, ds = p(r)\, \delta_{0m} \quad (m = 0, 1, 2, \ldots)$$

$$\ldots (5.13)$$

where δ_{0m} is the Kronecker's delta. Subtracting $(m+1)$th equation from $(m-1)$th equation, we get

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} G_n(s)\, G_m(s)\,[1+M(sh)]\, ds = p(r)\, \delta_{1m} \quad (m = 1, 2, \ldots). \quad \ldots (5.14)$$

It can be noted that the integrand is now symmetrical with respect to m and n. This is infinite set of linear simultaneous equations for determination of coefficients a_n. The method of numerically solving this set is outlined in section 7. Thus, we have determined the unknown function $N(sh)$ of (5.11) as a_n's are known.

6. Quantities of Physical Interest

We can now express displacements and stresses interms of known coefficients a_n. Especially, the displacement $w(r, 0)$ and $\sigma_{zz}(r, 0)$ are of physical importance and are given by

$$w(r, 0) = \begin{cases} 0, & r < r_i, \quad r > r_0 \\ (2/\pi b r_c) \sum_{n=1}^{\infty} a_n \sin n\phi, & r_i \le r \le r_0 \end{cases} \quad \ldots (6.1)$$

where

$$\phi = \cos^{-1}[(r_c^2 + b^2 - r^2)/2br_c]$$

$$\sigma_{zz}(r, 0) = 2\mu \left[\sum_{n=1}^{\infty} a_n \left\{ \int_0^{\infty} s\, G_n(s)\, J_0(sr)\, ds + g(r) \right\} - (1+\eta)\, \alpha_t T_0 \sum_{n=1}^{\infty} b_n h(r) \right] \quad \ldots (6.2)$$

where

$$g(r) = \int_0^{\infty} sM(sh)\, G_n(s)\, J_0(sr)\, ds \quad \ldots (6.3)$$

$$h(r) = \int_0^{\infty} \{L(sh)/(1 + e^{-2sh})\}\, Z_n(s)\, J_0(sr)\, ds. \quad \ldots (6.4)$$

Here, it is worth mentioning that the first term in the above equation coincides with the first term of (6.2) of Kumar and Padmshri[9]. The second and third terms in (6.2) show the effect of free boundaries. The integrals (6.3) and (6.4) are quickly convergent for large values of s. They shall not contribute to stress intensity factors. Using (3.6), eqn. (6.2) can be written as

$$\sigma_{zz}(r, 0) = 2\mu \left[\sum_{n=1}^{\infty} a_n \left\{ (-2n/br_c)(I_0^n + r\, \partial/\partial r\, I_0^n) + g(r) \right\} - (1+\eta)\, \alpha_t T_0 \sum_{n=1}^{\infty} b_n h(r) \right]. \quad \ldots (6.5)$$

The value of I_0^n shall differ when $r < r_i$ and $r > r_0$. We shall consider these cases separately.

Case $r < r_i$

For this region, we put $a = r_c$, $x = b$, $y = r$, $\mu = \nu = n$, $\alpha = 0$ in (3.7) to obtain

$$I_0^n = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\,\Gamma(n+1)\,r_c}\left(\frac{b}{r_c}\right)^n F(1/2,\ n+1/2,\ n+1;\ \sin^2\phi)\,F(1/2,\ n+1/2,\ 1,\ \sin^2\psi) \quad \ldots (6.6)$$

where

$$\begin{bmatrix}\psi\\ \phi\end{bmatrix} = \frac{1}{2}\left[\sin^{-1}\left(\frac{r+b}{r_c}\right) \pm \sin^{-1}\left(\frac{r-b}{r_c}\right)\right].$$

It is simple matter to derive

$$\frac{\partial}{\partial\phi}F(\beta,\eta,\gamma;\ \sin^2\phi) = \frac{\beta\eta}{\gamma}\sin 2\phi\ F(\beta+1,\ \eta+1,\ \gamma+1;\ \sin^2\phi)$$

and

$$\begin{bmatrix}\partial\psi/\partial\nu\\ \partial\phi/\partial\nu\end{bmatrix} = [R_1 \pm R_2]/2$$

where

$$R_1 = \{(r_0+r)(r_i-r)\}^{-1/2},\ R_2 = \{(r_0-r)(r_i+r)\}^{-1/2}.$$

Hence

$$\frac{\partial I_0^n}{\partial r} = \frac{\Gamma(n+3/2)}{4\Gamma(1/2)\,\Gamma(n+1)\,r_c}\left[\frac{b}{r_c}\right]^n \times\left[\frac{\sin 2\phi}{n+1}F\left(\frac{3}{2},\ n+\frac{3}{2},\ n+2;\ \sin^2\phi\right)F(1/2,\ n+1/2;\ 1;\ \sin^2\psi\ (R_1-R_2) + \sin 2\psi\ F(1/2,\ n+1/2,\ n+1;\ \sin^2\phi)\,F(3/2,\ n+3/2,\ 2;\ \sin^2\psi)\ (R_1+R_2)\right]. \quad \ldots (6.7)$$

Case $r > r_0$

For this region, we put $a = r$, $x = b$, $y = r_c$, $\alpha = \mu = n$, $\nu = 0$ to obtain

$$I_0^n = \frac{(-1)^n}{\pi r_0}\left[\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right]^2\left(\frac{br_c}{r_0^2}\right)^n F(n+1/2,\ n+1/2,\ n+1;\ \sin^2\phi) \times F(n+1/2,\ n+1/2,\ n+1;\ \sin^2\psi) \quad \ldots (6.8)$$

where

$$\begin{bmatrix}\psi\\ \phi\end{bmatrix} = [\sin^{-1}(r_0/r) \pm \sin^{-1}(r_i/r)]/2$$

and

$$\frac{\partial I_0^n}{\partial r} = \frac{(-1)^{n+1}}{2\pi (n+1)\, r_0^2}\left[\frac{\Gamma(n+3/2)}{\Gamma(n+1)}\right]^2\left(\frac{br_c}{r_0^2}\right)^n \left[\frac{4(n+1)}{n+1/2}\, F(n+1/2,\right.$$

$$n+1/2,\ n+1;\ \sin^2\phi)\ F(n+1/2,\ n+1/2,\ n+1;\ \sin^2\psi)$$

$$+ \sin 2\phi\, F\ (n+3/2,\ n+3/2,\ n+2;\ \sin^2\phi)\ F(n+1/2,$$

$$n+1/2,\ n+1;\ \sin^2\psi)\ (R_3 - R_4) + \sin 2\psi\ F(n+1/2,$$

$$n+1/2,\ n+1;\ \sin^2\phi)\ F(n+3/2,\ n+3/2,\ n+2;$$

$$\left.\sin^2\psi)\ (R_3 + R_4)\right]. \qquad \ldots (6.9)$$

where

$$R_3 = r_0\,(r^2 - r_0^2)^{-1/2}, \quad R_4 = r_i\,(r^2 - r_i^2)^{-1/2}.$$

The stress intensity factors at tips of the annular crack are defined by :

$$K_1 = \lim_{r \to r_i^-} \sqrt{2(r_i - r)}\ \sigma_{zz}(r, 0),\ r < r_i. \qquad \ldots (6.10)$$

$$K_2 = \lim_{r \to r_0^+} \sqrt{2(r - r_0)}\ \sigma_{zz}(r, 0),\ r > r_0. \qquad \ldots (6.11)$$

Thus, the two stress intensity factors can be expressed as :

$$K_1 = \frac{1}{\sqrt{r_c}} \sum_{n=1}^{\infty} a_n d_n \left[\frac{\sin 2\phi}{n+1}\ F(3/2,\ n+3/2,\ n+2;\ \sin^2\phi)\right.$$

$$F(1/2,\ n+1/2,\ 1;\ \sin^2\psi) + \sin 2\psi\ F(1/2,\ n+1/2;$$

$$\left.n+1;\ \sin^2\phi)\ F(3/2,\ n+3/2,\ 2;\ \sin^2\psi)\right] \ \ldots (6.12)$$

where

$$\begin{bmatrix}\psi\\ \phi\end{bmatrix} = \pi/4 \pm (1/2)\sin^{-1}(1 - 2b/r_c)$$

and

$$K_2 = (1/\sqrt{r_0}) \sum_{n=1}^{\infty} a_n d_n' \left[\sin 2\phi\, F(n+3/2, n+3/2, n+2;\ \sin^2\phi)\right.$$

$$F(n+1/2,\ n+1/2,\ n+1;\ \sin^2\psi) + \sin 2\psi\ F(n+1/2,$$

$$\left.n+1/2,\ n+1;\ \sin^2\phi)\ F(n+3/2,\ n+3/2,\ n+2;\ \sin^2\psi)\right]$$

$$\ldots (6.13)$$

where

$$\begin{bmatrix} \psi \\ \phi \end{bmatrix} = [\pi \pm \sin^{-1}(r_i/r_0)]/2.$$

The values of d_n and d_n' are given by

$$d = \frac{n\mu\,\Gamma(n+3/2)\,r_i}{\Gamma(1/2)\,\Gamma(n+1)\,br_c^2}\left(\frac{b}{\gamma_c}\right)^n$$

$$d_n' = \frac{(-1)^{n+1}\,n\mu}{\pi(n+1)\,r_0^2} \times \left[\frac{\Gamma(n+3/2)}{\Gamma(n+1)}\right]^2 \left[\frac{br_c}{r_0}\right]^n.$$

7. Numerical Solution

This section outlines the method of solving infinite set of linear simultaneous equations (5.14). A general element of the coefficient matrix may be written as

$$A_{nm} = A_{mn} = B_{mn} + C_{mn} \qquad \ldots (7.1)$$

where

$$B_{mn} = \int_0^\infty G_n(s)\,G_m(s)\,ds \qquad \ldots (7.2)$$

$$C_{mn} = \int_0^\infty G_n(s)\,G_m(s)\,M(sh)\,ds \qquad \ldots (7.3)$$

and $M(sh)$ is given by (5.8). The integrand in C_{mn} is quickly convergent for large values of s and hence evaluated by Gauss-Laguerre integration formula. The term B_{mn} shall be evaluated by the method of Kumar and Padmshri[9] because this term is same as in (7.1) of Kumar and Padmshri[9]. In short, we can say that

$$B_{mn} = \int_0^\lambda G_n(s)\,G_m(s)\,ds + B_{mn}' \qquad \ldots (7.4)$$

where

$$B_{mn}' = \int_\lambda^\infty G_n(s)\,G_m(s)\,ds = \frac{4mn}{\pi^2 b^3 r_c^3}\,[r_i^2\{\lambda^{-1}\sin^2\lambda r_i - r_i\,si(2\lambda r_i)\}$$

$$+ \{(-1)^m + (-1)^n\} \times \{\lambda^{-1}\sin\lambda r_i\cos\lambda r_0 - r_c\,ci(2\lambda r_c) + b\,ci(2\lambda b)\}\; r_0 r_i$$

$$+ (-1)^{m+n} r_0^2\{\lambda^{-1}\cos^2\lambda r_0 + r_0\,si(2\lambda r_0)] \qquad \ldots (7.5)$$

where

$$si(x) = \int_\infty^x t^{-1}\sin t\,dt, \quad ci(x) = \int_\infty^x t^{-1}\cos t\,dt.$$

This would completely determine coefficients A_{mn}. The infinite set of simultaneous equations (5.14) is solved by considering 15 equations in 15 unknowns. However, it is ntoiced that the values of a_n become too small when $n \geq 10$ and hence first ten roots are considered. The calculations are performed for $\lambda = 100$, $\eta = 0.290$, $\alpha_t = 18.80 \times 10^{-6}$, $T_0 = 100°C$. The value of outer radius of annular crack r_0 is fixed equal to 1 and r_i, the inner radius is varied from 0.2 to 1.0 in steps of 0.2. The value of h is varied from 1 to 5 in steps of 1. Figure 2 shows variation of two normal stress intensity factors K_1/K_1^∞ and K_2/K_2^∞ with ri/r_0. Figure 3 shows variation of intensity factors K_1/K_1^∞ and K_2/K_2^∞ with h. It should be noted that the two intensity factors take almost the same values of infinite solid (see Kumar

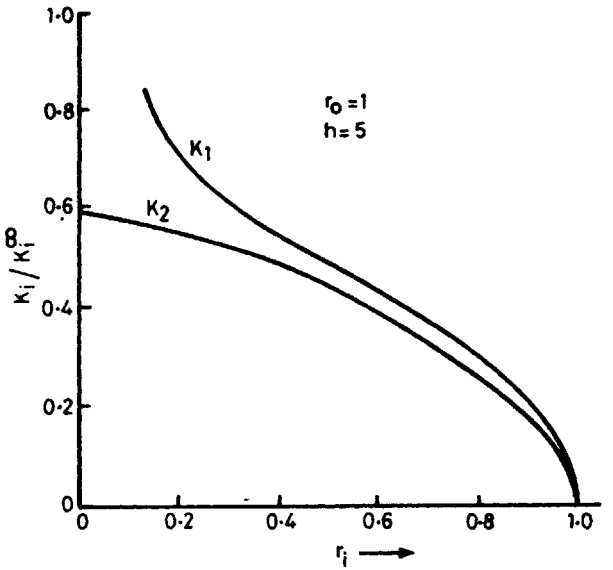

FIG. 2. Variation of intensity factors with r_i/r_0.

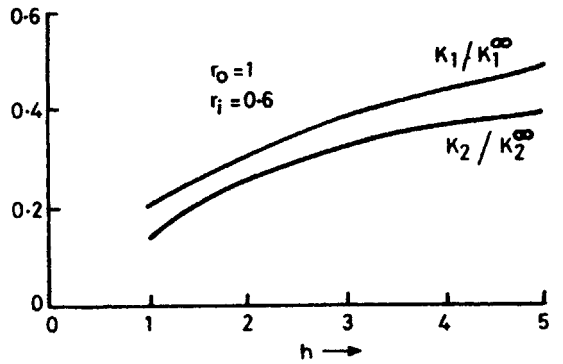

FIG. 3. Variation of intensity factors with h.

and Padmshri[9]) when $h = 5$. The variations are similar in nature as shown in Fig. 4 of Mori *et al.*[10].

It must be noted that infinite set of simultaneous equations (4.8) are solved by the method explained above.

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