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NONLINEAR HYBRID CONTRACTIONS ON MENGER AND UNIFORM SPACES

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In this paper we show the existence of solutions of functional equations $Tx \in Px$, $Sx = Tx \in Px \cap Qx$ and $x = Tx \in Px \cap Qx$ under certain hybrid contraction conditions, where S, T and P, Q are single- and multi-valued mappings on Menger spaces. Some of the results are extended to uniform spaces.

1. INTRODUCTION

Following the Banach contraction principle, Nadler[27] introduced the concept of multivalued contraction mappings and established the multivalued contraction principle (see Theorem 2.1 below which is usually called Nadler's multivalued contraction principle). Subsequently, a number of fixed point theorems were obtained for multivalued mappings on metric, vector and uniform spaces (see, for instance, under references [1], [3], [6], [13], [15], [17], [19], [20], [22]-[26], [28]-[30], [33], [34], [40], [43] and [45]). Egbert[10] introduced the concept of probabilistic diameter of a set, and extended the definition of Hausdorff metric to probabilistic spaces.

The theory of multivalued mappings has wide applications to game theory, optimal control theory and mathematical economics. Fixed points of multivalued mappings are used for the study of random operator equations and their applications[2, 4, 16, 18, 32]. A short review of results concerning multivalued mappings in metrizable spaces was given by Hadžić[16]. She[14-18] and Pai and Veeramani[29] proved certain results regarding fixed points of multivalued mappings in PM-spaces. Singh and Pant[41] (see also Pant[30]) established a coincidence theorem for hybrid contractions in Menger spaces. In fact, in contrast to Hadžić[14-16] and Pai and Veeramani[29], Singh and Pant[41] developed an elegant theory for multivalued contractions on Menger spaces. It appears that in comparison to Hadžić[14] and Pai and Veeramani[29], their work is akin to the corresponding work on metric spaces. Tan[42, 43] utilizing the known fixed point theorems for multivalued mappings on uniform spaces, obtained similar results on PM-spaces. In a recent formulation, Tan[44] has established fixed point theorems for multivalued contractions on Menger spaces. His[44] Corollary 2.3 is the Menger space version of a slightly generalized form of Nadler's contraction principle for multivalued mappings on metric spaces.

The intent of this paper is to investigate conditions under which single- and multi-valued mappings on Menger and uniform spaces have coincidences and fixed points. Some of the earlier results/variants[5, 6-9, 12, 17-19, 21, 25, 27, 28, 30, 33, 34, 38-41] may be derived (exactly or slightly under different conditions) from the results of this paper.

2. Preliminaries

The following are the notations and definitions to be used in the sequel.

Let (M, d) be a metric space, $CB(M)$, the set of nonempty closed bounded subsets of M, and H, the Hausdorff metric on $CB(M)$ induced by the metric d.

$P : M \to CB(M)$ is a multivalued contraction mapping (m.v.c.m.) on M if there exists a constant $\alpha \in (0, 1)$ such that

(m_1) $H(Pu, Pv) \leq \alpha\, d(u, v)$

for all u, v in M.

Theorem 2.1 (Nadler[27]) — Let (M, d) be a complete metric space. If $P : M \to CB(M)$ is an m.v.c.m., then P has a fixed point, i.e., there exists a point z in M such that $z \in Pz$.

Let $(X, \mathcal{F}, t)$ be a Menger space[7, 9, 35, 36, 40]. The value of $\mathcal{F}$ at $(u, v) \in X \times X$ is represented by $F(u, v)$. Let $H^*(x)$ denote a distribution function defined as

$$H^*(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

For any nonempty subsets A, B of X, distribution functions G and L are defined as follows[40] :

$$G(A, B; x) = \operatorname*{lub}_{k<x} \left\{ \operatorname*{glb}_{\substack{u \in A \\ v \in B}} F(u, v; k) \right\} \quad \text{... (2.1.1)}$$

$$L(A, B; x) = \operatorname*{lub}_{k<x} \left\{ \operatorname*{lub}_{\substack{u \in A \\ v \in B}} F(u, v; k) \right\} \quad \text{... (2.1.2)}$$

When $A = \{u\}$, $L(u, B; x)$ denotes the probability that ordinary distance between the point u and the set B is less than x. Also, for $v \in B$,

$$F(u, v; x) \leq L(A, B; x),\ x > 0.$$

The function E defined by

$$E(A, B; x) = \underset{k<x}{\text{lub}}\ t\left\{\underset{u\in A}{\text{glb}}\left(\underset{v\in B}{\text{lub}}\ F(u, v; k)\right),\ \underset{v\in B}{\text{glb}}\left(\underset{u\in A}{\text{lub}}\ F(u, v; k)\right)\right\} \quad \text{... (2.1.3)}$$

is called the probabilistic distance between A and B (Egbert[10], Def. 5), and is a distribution function[10]. The class of such functions is denoted by ξ.

For any subsets A, B of X,

$$G(A, B; x) \le E(A, B; x) \le L(A, B; x).$$

With t = min and A, B singletons we have

$$E(A, B) = L(A, B) = G(A, B).$$

Throughout this paper, K stands for an arbitrary nonempty set. $C(X)$ stands for the (nonempty) closed subsets of X.

If E is defined for $A, B \in C(X)$ then Egbert[10] has shown that $(C(X), \xi, t)$ is a Menger space.

The value of ξ at $A, B \in C(X)$, as usual, will be denoted by $E(A, B)$. The space $(C(X), \xi, t)$ may be called Egbert-Hausdorff Menger space or simply EHM-space induced by $(X, \mathcal{F}, t)$.

Let $P : K \to X$ and $Q : K \to C(X)$. Then P and Q are said to commute if for each $u \in K$,

$$P(Q(u)) = PQ(u) \subseteq QP(u) = Q(P(u)).$$

The following lemmas form the essential part of the proof of our results.

Lemma 2.2[40] — Let A, B be in $C(X)$. Then for all u in A and for some h, b in $(0, 1)$, there exists a v in B such that

$$F(u, v; h^{-b} x) \ge E(A, B; x) \text{ for } x > 0.$$

Lemma 2.3[40] — Let A be in $C(X)$ and b in $(0, 1)$. Then for every u in A, there exists a v in A such that for $x > 0$,

$$G(u, A; b^{-1} x) \ge F(u, v; x)$$

and

$$E(u, A; b^{-1} x) \ge F(u, v; x).$$

Lemma 2.4[40] — Let A, B and C be nonempty subsets of X. Then for a fixed u in A and any $v \in C$,

$$L(u, B; x + y) \ge t\{F(u, v; x),\ L(C, B; y)\} \text{ for } x > 0,\ y > 0.$$

Lemma 2.5[40] — Let A, B and C be nonempty subsets of X. Then for a fixed u in A and any v in C,

$$G(u, B; x + y) \ge t\{F(u, v; x),\ G(v, B; y)\} \text{ for } x > 0,\ y > 0.$$

3. Hybrid Contractions

Let f be a mapping on a metric space M and $P : M \to CB(M)$. If there exists a number $k \in (0, 1)$ such that

(H_1) $H(Px, Py) \leq kd(fx, fy)$

for all x, y in M, then (H_1) is called a hybrid contraction. The study of hybrid (type) contractions was initiated independently by Singh and Kulshreshtha[38] and Subrahmanyam (cf. Sastry *et al.*[34]; see also Itoh and Takahashi[20]). Recall that if f is the identity mapping on M, then (H_1) and (m_1) are the same. The following example shows that P and f satisfying (H_1) need not have a common fixed point.

Example 3.1[28] — Let $X = [0, \infty)$, $Tx = [1 + x, \infty)$ and $fx = 2x$ for $x \in X$. Clearly (H_1) holds with $\alpha \in (1/2, 1)$, and f, T do not have a common fixed point. However f and T have coincidences. Evidently $fz \in Tz$ for $z \geq 1$.

Due to the possibility of wide applications of coincidence theorems for single- and multi-valued mappings, the basic result established by Singh and Kulshreshtha[38] drew a good attention, and subsequently a number of mathematicians established coincidence theorems for contractive type conditions involving single- and multi-valued mappings on metric spaces[3, 11, 21, 25, 28, 31, 37, 40]. However, the study of probabilistic hybrid contractions appears to have been initiated by Singh and Pant[40]. Infact they proved the following :

Theorem 3.2[40] — Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$, and let P and Q be mappings from an arbitrary (nonempty) set K to $C(X)$. If there exist a mapping $T : K \to X$ and a positive number $h < 1$ such that $T(K)$ is a complete subspace of X with $P(K) \cup Q(K) \subset T(K)$, and

$$E(Pu, Qv; h\varepsilon) \geq \min \{F(Tu, Tv; \varepsilon),\ L(Tu, Pu; \varepsilon),$$
$$L(Tv, Qv; \varepsilon),\ L(Tu, Qv; 2\varepsilon),\ L(Tv, Pu; 2\varepsilon)\}$$

for all u, v in K and $\varepsilon > 0$; then P, Q and T have a coincidence, i.e., there exists a point z in K such that $Tz \in (Pz \cap Qz)$.

Coincidence Theorems for Three Mappings

Theorem 3.3 — Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. Let P and Q be multivalued mappings from X to $C(X)$. Further, let $T : X \to X$ be such that T^2 is continuous. Let T commute with each of P and Q, and $P(X) \cup Q(X) \subset T(X)$. If there exists a constant $h \in (0, 1)$ such that

$$E(Pp, Qq; h\varepsilon) \geq \min \{F(Tp, Tq; \varepsilon),\ L(Tp, Pp; \varepsilon)$$
$$L(Tq, Qq; \varepsilon),\ L(Tp, Qq; 2\varepsilon),\ L(Tq, Pp; 2\varepsilon)\} \quad \ldots (3.3.1)$$

for all $p, q \in X$ and $\varepsilon > 0$, then P, Q and T have a coincidence.

PROOF : It may be completed generally following the proof of Theorem 3.2 above (see Singh and Pant[40], Proof of Theorem 1).

(Note that Theorem 3.2 is a special case of Theorem 3.8 below. Further, conditions on t in Theorem 3.3 imply that $t(x, y) = \min\{x, y\}$).

The following is a variant of Theorem 3.3.

Theorem 3.4 — Let $(X, \mathcal{F}, t)$ be a complete Menger space and $P, Q : X \to C(X)$. Further, let $T : X \to X$ be such that T^2 is continuous and T commutes with each of P and Q. If, for a natural number r, (3.3.1) is satisfied with T replaced by T^r, then P, Q and T have a coincidence.

Corollary 3.5 — Let (M, d) be a complete metric space and $P, Q : M \to C(M)$, $T : M \to M$ such that T^2 is continuous and T commutes with each of P and Q. If $P(M) \cup Q(M) \subset T(M)$ and

$$H(Pp, Qq) \leq h \max\{d(Tp, Tq), d(Tp, Pp), d(Tq, Qq), 1/2\, d(Tp, Qq), 1/2\, d(Tq, Pp)\} \quad \text{... (3.5.1)}$$

holds for some $h \in (0, 1)$ and for all $p, q \in M$, then P, Q and T have a coincidence.

PROOF : First we define a mapping $\mathcal{F} : M \times M \to \mathcal{L}$ as follows :

$$F(x, y; t) = H^*(t - d(x, y)) \quad \text{... (3.5.2)}$$

for all $x, y \in M$, $t > 0$. Then $(M, \mathcal{F}, \min)$ with t-norm = 'min' is a complete Menger space induced by the complete metric space M.

For $x \in M$ and $A \in C(M)$, we define a probabilistic distance $L(x, A)$ as follows :

$$L(x, A; t) = H^*(t - d(x, A)).$$

It is easy to prove that the Menger-Hausdorff metric E induced by $\mathcal{F}$ (defined by (3.5.2)) has the following form :

$$E(A, B; t) = H^*(t - H(A, B)),\ A, B \in C(M).$$

By (3.5.1), for any $p, q \in M$, $t > 0$ we have

$$E(Pp, Qq; t) = H^*(t - H(Pq, Qq))$$
$$\geq H^*(t - h \max\{d(Tp, Tq), d(Tp, Pp), d(Tq, Qq), 1/2\ d(Tp, Qq), 1/2\ d(Tq, Pp)\}$$
$$= H^*(t/h - \max\{d(Tp, Tq), d(Tp, Pp), d(Tq, Qq), 1/2\ d(Tp, Qq), 1/2\ d(Tq, Pp)\}$$
$$= \min\{F(Tp, Tq; t/h), F(Tp, Pp; t/h), F(Tq, Qq; t/h), F(Tp, Qq; 2t/h), F(Tq, Pp; 2t/h)\},\ t > 0.$$

Therefore all the conditions of Theorem 3.3 are satisfied, and Corollary 3.5 follows from Theorem 3.3 immediately.

Corollary 3.6 — Let X be a complete Menger space and $P, Q, T : X \to X$ be such that T^2 is continuous, T commutes with each of P and Q, and $P(X) \cup Q(X) \subset T(X)$. If there exists a constant $h \in (0, 1)$ such that

$$F(Pp, Qq; h\varepsilon) \geq \min \{F(Tp, Tq; \varepsilon), F(Tp, Pp; \varepsilon),$$

$$F(Tq, Qq; \varepsilon), F(Tp, Qq; 2\varepsilon), F(Pp, Tq; 2\varepsilon)\} \quad \ldots (3.6.1)$$

for all $p, q \in X$ and $\varepsilon > 0$, then P, Q and T have a unique common fixed point.

PROOF : Theorem 3.3 with $P, Q, T : X \to X$, immediately proves that P, Q and T have a coincidence point in X. If Tz is a coincidence point of P, Q and T, i.e., $T(Tz) = P(Tz) = Q(Tz)$, then using (3.6.1) it can be easily proved that Tz is the unique common fixed point of P, Q and T.

The above result is an extension of Dedeic and Sarapa[8].

Theorem 3.7 — Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. Let $P, Q : X \to C(X)$ such that

$$G(Pp, Qq; h\varepsilon) \geq \min \{F(p, q; \varepsilon), E(p, Pp; \varepsilon),$$

$$E(q, Qq; \varepsilon), E(p, Qq; 2\varepsilon), E(q, Pp; 2\varepsilon)\} \ldots (3.7.1)$$

for all $p, q \in X$, $\varepsilon > 0$ and some h in $(0, 1)$. Then there exists a unique z in X such that $\{z\} = Pz \cap Qz$.

PROOF : It may be proved using Lemma 2.3 and Corollary 3.6 with $Tx = x$ for each x in X.

A General Coincidence Theorem

Theorem 3.8 — Let $(X, \mathcal{F}, t)$ be a Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$, and let P, Q be multivalued mappings from an arbitrary (nonempty) set K to $C(X)$. If there exist mappings $S, T : K \to X$ and a positive number $h < 1$ such that

$$Q(K) \subset S(K), \; P(K) \subset T(K)$$

and

$$E(Pu, Qv; hx) \geq \min \{F(Su, Tv; x), L(Su, Pu; x), L(Tv, Qv; x),$$

$$L(Tv, Pu; 2x), L(Su, Qv; 2x)\}, \; x > 0, \; u, v \text{ in } K; \quad \ldots (3.8.1)$$

then :

(a) if $S(K)$ (resp. $(T(K))$ is a complete subspace of X, then P and S (resp. Q and T) have a coincidence; and

(b) if $S(K) \cap T(K)$ is a complete subspace of X then

(i) P and S have a coincidence

(ii) Q and T have a coincidence.

PROOF : Pick u_0 in K. Choose a point u_1 in K such that $v_1 = Tu_1 \in Pu_0$. We can do this since $P(K) \subset T(K)$. Further, since $Q(K) \subset S(K)$ we can choose $v_2 = Su_2 \in Qu_1$ such that

$$F(v_1, v_2; h^{-b} x) \geq E(Pu_0, Qu_1; x)$$

for some b in (0, 1). This is possible by Lemma 2.2 for $x > 0$. Similarly, choose

$$v_3 = Tu_3 \in Pu_2$$

such that $\quad F(v_2, v_3; h^{-b} x) \geq E(Qu_1, Pu_2; x), \quad x > 0.$

In general, we choose

$$v_{2n+1} = Tu_{2n+1} \in Pu_{2n}$$

such that $\quad F(v_{2n}, v_{2n+1}; h^{-b} x) \geq E(Pu_{2n}, Qu_{2n-1}; x), \quad x > 0;$

and

$$v_{2n+2} = Su_{2n+2} \in Qu_{2n+1}$$

such that $\quad F(v_{2n+1}, v_{2n+2}; h^{-b} x) \geq E(Pu_{2n}, Qu_{2n+1}; x), \quad x > 0.$

Now by (3.8.1),

$$\begin{aligned} F(v_{2n}, v_{2n+1}; h^{1-b} x) &\geq E(Pu_{2n}, Qu_{2n-1}; hx) \\ &\geq \min \{F(v_{2n}, v_{2n-1}; x), L(v_{2n}, Pu_{2n}; x), \\ &\qquad L(v_{2n-1}, Qu_{2n-1}; x), L(v_{2n-1}, Pu_{2n}; 2x), \\ &\qquad L(v_{2n}, Qu_{2n-1}; 2x)\} \\ &\geq \min \{F(v_{2n}, v_{2n-1}; x), F(v_{2n}, v_{2n+1}; x), \\ &\qquad F(v_{2n-1}, v_{2n}; x), F(v_{2n-1}, v_{2n+1}; 2x), F(v_{2n}, v_{2n}; 2x)\}. \end{aligned}$$

Since $\quad F(v_{2n-1}, v_{2n+1}; 2x) \geq \min \{F(v_{2n-1}, v_{2n}; x), F(v_{2n}, v_{2n+1}; x)\},$

we have $\quad F(v_{2n}, v_{2n+1}; h' x) \geq F(v_{2n-1}, v_{2n}; x), \quad \ldots (3.8.2)$

where $\quad h' = h^{1-b} < 1.$

Similarly

$$F(v_{2n+1}, v_{2n+2}; h' x) \geq F(v_{2n}, v_{2n+1}; x). \quad \ldots (3.8.3)$$

In view of (3.8.2) and (3.8.3), we have

$$F(v_n, v_{n+1}; h' x) \geq F(v_{n-1}, v_n; x), \; n = 1, 2, \ldots.$$

Since $h' < 1$, $\{v_n\}$ is a Cauchy sequence in $S(K)$. If $S(K)$ is a complete subspace of X then $\{v_n\}$ converges to a point z in $S(K)$. We now show that P and S have a coincidence.

Let w be a point in $S^{-1}z$. So, $Sw = z$.

Let $U_{Sw}(\varepsilon, \lambda)$ be a neighbourhood of Sw.

Since $v_n \to Sw$, for $\varepsilon > 0$, $\lambda > 0$ there exists $N = N(\varepsilon, \lambda)$ such that

$$F(Sw, v_{2n+2};\ (1-h)\,\varepsilon/2) > 1-\lambda \text{ for all } n \geq N. \qquad \text{... (3.8.4)}$$

Now by Lemma 2.4,

$$L(Sw, Pw; \varepsilon) \geq \min\ \{F(Sw, v_{2n+2};\ (1-h)\,\varepsilon/2),$$
$$L(Qu_{2n+1}, Pw; (1+h)\,\varepsilon/2)\}. \qquad \text{... (3.8.5)}$$

By (3.8.1),

$$L(Pw, v_{2n+2}; \varepsilon) \geq E(Pw, Qu_{2n+1}; \varepsilon)$$
$$\geq \min\ F(Sw, Tu_{2n+1}; \varepsilon/h),\ L(Sw, Pw, \varepsilon/h),$$
$$L(Tu_{2n+1}, Qu_{2n+1}; \varepsilon/h),\ L(Tu_{2n+1}, Pw; 2\varepsilon/h),$$
$$L(Sw, Qu_{2n+1}; 2\varepsilon/h)\}$$
$$\geq \min\ \{F(Sw, v_{2n+1};\ (1-h)\,\varepsilon/2),\ F(Sw, v_{2n+2};\ (1-h)\,\varepsilon/2h),$$
$$L(Pw, Qu_{2n+1};\ (1+h)\,\varepsilon/2h),\ F(v_{2n+1}, v_{2n+2};\ \varepsilon/h),$$
$$L(Pw, Qu_{2n+1};\ (1+h)\,\varepsilon/h),\ F(v_{2n+1}, v_{2n+2};\ (1-h)\,\varepsilon/h),$$
$$F(Sw, v_{2n+2};\ (1-h)\,\varepsilon/2)\}$$
$$> 1-\lambda \text{ for all } n \geq N \text{ (using 3.8.4)}$$

that is,

$$L(Sw,\ Pw;\ \varepsilon) > 1-\lambda \text{ for all } n \geq N.$$

Hence $Sw \in Pw$. Similarly, if $T(K)$ is a complete subspace of X then Q and T have a coincidence. Now the part (b) is evident.

Corollary 3.9 — Let (M, d) be a metric space and let $P, Q : M \to C(M)$. If there exist mappings $S, T : M \to M$ and a positive number $h < 1$ such that

$$H(Px,\ Qy) \leq h \max\ \{d(Sx,\ Ty),\ d(Px,\ Sx),$$
$$d(Qy,\ Ty),\ 1/2\ d(Px,\ Ty),\ 1/2\ d(Qy,\ Sx)\} \qquad \text{... (3.9.1)}$$

for all $x, y \in M$, then :

(a) if $Q(M) \subset S(M)$, $P(M) \subset T(M)$ and $S(M)$ (resp. $T(M)$) is a complete subspace of M, then P and S (resp. Q and T) have a coincidence.

(b) if $S(M) \cap T(M)$ is a complete subspace of M, then

(i) P and S have a coincidence, and

(ii) Q and T have a coincidence.

PROOF : Defining the Menger-Hausdorff metric E as in Corollary 3.5, it can be easily proved that (3.9.1) implies (3.8.1). Hence the result follows from Theorem 3.8.

Corollary 3.10 — Let $(X, \mathcal{F}, t)$ be a Menger space where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$ and P be a multivalued mapping from K to $C(X)$. If there exist a mapping $T : K \to X$ and a positive number $h < 1$ such that

$$E(Pu, Pv; hx) \geq F(Tu, Tv; x) \quad \text{... (3.10.1)}$$

for all u, v in K and $x > 0$; and if $P(K) \subset T(K)$ and $T(K)$ is a complete subspace of X, then P and T have a coincidence.

PROOF : Evident, since (3.10.1) $\Rightarrow$ (3.8.1).

This result extends and unifies certain coincidence theorems proved in Goebel[12] and Naimpally *et al.*[28] and the Banach contraction principle for m.v.c.m. in metric spaces (cf. Nadler[27]). A single-valued version of results (3.8)-(3.10) appears in Singh and Pant[39].

Recall that Theorem 3.8 with $S = T$ is Theorem 3.2.

The following result is immediate from Theorem 3.3 when T is the identity mapping on X.

Corollary 3.11 — Let $(X, \mathcal{F}, t)$ be a complete Menger space, where t is continuous and satisfies $t(x, x) \geq x$ for every $x \in [0, 1]$. Let P be a multivalued mapping from X to $C(X)$. If there exists a constant $h \in (0, 1)$ such that

$$E(Pp, Pq; h\varepsilon) \geq \min \{F(p, q; \varepsilon), L(p, Pp; \varepsilon), L(q, Pq; \varepsilon), L(p, Pq, 2\varepsilon), L(q, Pp; 2\varepsilon)\} \quad \text{... (3.11.1)}$$

for all $p, q \in K$ and $\varepsilon > 0$. Then P has a fixed point.

Definition 3.12[37] — Let M be a metric space. Then $T : M \to M$ and $S : M \to C(M)$ are weakly commuting at $z \in M$ iff $H(STz, TSz) \leq D(Tz, Sz)$. The mappings S and T are weakly commuting on M if they weakly commute at every point of M.

We now introduce the following definition of weak commutativity on Menger spaces :

Definition 3.13 — Let $(X, \mathcal{F}, t)$ be a Menger space. Then $T : X \to X$ and $S : X \to C(X)$ are said to commute weakly at $z \in X$ iff $E(STz, TSz; x) \geq L(Sz, Tz; x)$, $x > 0$.

They are said to weakly commute on X if they do so at every point of the space.

Corollary 3.14 — Theorem 3.2. Further, if $K = X$, and T weakly commutes with each of P and Q at a coincidence point w, and Tw is a fixed point of T; then P, Q and T have a common fixed point, i.e., $T(Tw) = Tw \in P(Tw) \cap Q(Tw)$.

PROOF : The coincidence part of this corollary follows from Theorem 3.2 or Theorem 3.8. If, for some w in X, $Tw \in Pw \cap Qw$ and $T(Tw) = Tw$ then $Tw = TTw \in TPw$. By the weak commutativity of T and P, $TPw = PTw$, and $Tw \in PTw$. Similarly Tw is a fixed point of Q.

4. Extension to Uniform Spaces

Many important results regarding the existence of fixed points for multivalued mappings in uniform spaces have been obtained over the last two decades.

Cain and Kasriel[5] (see also Constantin and Istrǎtescu[7] and Hicks[19]) have shown that a connection between PM-spaces and uniform spaces can be established using the probabilistic metric $\mathcal{F}$, which is very clear from the following two propositions.

Proposition 4.1[5] (see also Constantin and Istrǎţescu[7], page 66) — Let $(X, \mathcal{F}, \min)$ be a Menger space and $d_\alpha : X \times X \to R$ be the family of functions defined by

$$d_\alpha(p, q) = \sup\{t : F(p, q\,; t) \leq 1 - \alpha\} \qquad \text{... (4.1.1)}$$

$\alpha \in (0, 1)$. Then the family $\{d_\alpha\}_{\alpha \in (0, 1)}$ is a family of pseudo-metrics associated with the probabilistic metric $\mathcal{F}$ which satisfies the following properties :

(i) $d_\alpha(p, q)$ is finite and for all $\alpha \in (0, 1)$,

$$d_\alpha(p, q) = 0 \text{ if and only if } p = q;$$

(ii) for each $(p, q) \in X \times X$, $d_\alpha(p, q)$ is a left continuous non-increasing function of α such that

$$F(p, q; (d_\alpha(p, q))) \leq 1 - \alpha;$$

(iii) the topology on X generated by $\{d_\alpha\}_{\alpha \in (0, 1)}$ is the same as the topology induced by $\mathcal{F}$.

Since for fixed $p, q \in X$, $d_\alpha(p, q)$ is a non-increasing function of α, it follows that for any $\delta > 0$, the family $\{d_\alpha\}_{\alpha \in (0, \delta)}$ generates the same topology as that generated by the entire family $\{d_\alpha\}_{\alpha \in (0, 1)}$.

Cain and Kasriel[5] have also shown that there exists a probabilistic metric $\mathcal{F}$ on X such that $\{d_\alpha\}_{\alpha \in (0, 1)}$ is the family of pseudometrics associated with $\mathcal{F}$.

Proposition 4.2 [5, 7] — Let X be a Hausdorff space with a topology generated by a family of pseudometrics $\{d_\alpha\}_{\alpha \in (0, 1)}$ such that, for $p, q \in X, d_\alpha(p, q)$ is a non-increasing left continuous function of α with $d_\alpha(p, q) = 0$ if and only if $p = q$. Then there exists a probabilistic metric $\mathcal{F}$ on X such that $\{d_\alpha\}_{\alpha \in (0, 1)}$ is the family of pseudometrics associated with it.

Defining $\mathcal{F}(p, q)$ as

$$F(p, q; t) = \{1 - \sup\{\alpha \in (0, 1) : d_\alpha(p, q) \geq t\}, \quad \text{if } \{\alpha \in (0, 1) : d_\alpha(p, q) \geq t\} \neq \phi$$
$$1, \quad \text{if } d_\alpha(p, q) < t \text{ for each } \alpha \in (0, 1)\}, \qquad \text{... (4.2.1)}$$

where p, q are two fixed elements in X and $t > 0$, the above propositions can be easily verified.

Remark 4.2[7] : It is easy to see that $(X, \mathcal{F}, \min)$ is a Menger space, and the family of pseudometrics associated with the probabilistic metric $\mathcal{F}$ is precisely the family $\{d_\alpha\}_{\alpha \in (0,1)}$ where d_α is defined by (4.1.1).

If $\mathcal{P} = \{d_\alpha\}$ is a family of pseudometrics on X. The uniformity $\mathcal{U}$ generated by $\mathcal{P}$ is constructed by taking as a subbase of all sets in $(X \times X)$ of the form

$$\beta = V_{\alpha,\varepsilon} = \{(p, q) : d_\alpha(p,q) < \varepsilon\}$$

where $d_\alpha \in \mathcal{P}$ and $\varepsilon > 0$. We may assume β itself to be a base by adjoining finite intersection of members of β if necessary. The corresponding family of pseudometrics is called an augmented associated family for $\mathcal{U}$ and is denoted by $\mathcal{P}^*$. The topology τ determined by this uniformity has all d_α-spheres as a subbase. Also by assuming that for $p, q \in X, p \neq q$, there exists at least one $d_\alpha \in \mathcal{P}$ for which $d_\alpha(p,q) > 0$, the space (x, y) necessarily becomes Hausdorff.

Making use of Propositions 4.1 and 4.2 many, interesting results can be converted from uniform spaces to PM-spaces and conversely, from PM-spaces to uniform spaces.

Let $(X, \mathcal{U})$ be a uniform space and let $U \in \mathcal{U}$ be arbitrary. For each subset A of X define $U[A] = \{y \in X : (x,y) \in U$ for some $X \in A\}$. The Hausdorff uniformity $2^{\mathcal{U}}$ on 2^X (the collection of nonempty closed subsets of X) is defined by the base $2^\beta = \{\widetilde{U} : U \in \mathcal{U}\}$ when

$$\widetilde{U} = \{(A, B) : A, B \in 2^X, A \subseteq \bigcup [B], B \subseteq \bigcup [A]\}.$$

The augmented associated family $\mathcal{P}^*$ for $\mathcal{U}$ induces a uniformity $\mathcal{U}^*$ on 2^X defined by the base $\beta^* = \{V^*_{\alpha,\varepsilon}\}$ where

$$V^*_{\alpha,\varepsilon} = \{(A, B) : A, B \in 2^X, H_\alpha(A,B) < \varepsilon\}$$

and $$H_\alpha(A,B) = \max\{\sup d_\alpha(a,B) : a \in A, \sup d_\alpha(A,b) : b \in B\}.$$

Then the uniformities $2^{\mathcal{U}}$ and $\mathcal{U}^*$ on 2^X are uniformly isomorphic. The space $(2^X, \mathcal{U}^*)$ is thus a uniform space, called the hyper space of $(X, \mathcal{U})$.

The following results extend Theorem 3.3 and Theorem 3.8 to uniform spaces.

Theorem 4.3 — Let X be a sequentially complete Hausdorff uniform space and P, Q multivalued mappings from X to 2^X (the collection of nonempty closed subsets of X). Further, suppose that $T : X \to X$ be such that T^2 is continuous. Let T commute with each of P and Q and $P(X) \bigcup Q(X) \subset T(X)$. If there exists a constant $h_\lambda \in (0, 1)$ such that for all $p, q \in X$ and $d_\lambda \in \mathcal{P}$ ($\lambda \in I, I$ being indexing set),

$$H_\lambda(Pp, Qq) \le h_\lambda \max\{d_\lambda(Tp, Tq), d_\lambda(Tp, Pp), d_\lambda(Tq, Qq),$$
$$1/2\ d_\lambda(Tp, Qq), 1/2\, d_\lambda(Tq, Pp)\} \quad \text{... (4.3.1)}$$

then P, Q and T have a coincidence.

PROOF : Defining $\mathcal{F}$ by (4.2.1) it is easy to show that $(X, \mathcal{F}, \min)$ is a Menger space and the family of pseudometrics associated with $\mathcal{F}$ is given by

$$d_\alpha(p, q) = \sup_{\alpha \in (0,1)} \{t : F(p, q; t) \le 1 - \alpha\}.$$

We must show that (4.3.1) implies (3.3.1), that is, mappings P, Q and T satisfying (4.3.1) also satisfy (3.3.1). Assume that P, Q and T satisfy (4.3.1) for all p, q in X and fail to satisfy (3.3.1) for some p, q in X.

Then

$$E(Pp, Qq; h\varepsilon) < \min\{F(Tp, Tq; \varepsilon), L(Tp, Pp; \varepsilon), L(Tq, Qq; \varepsilon),$$
$$L(Tp, Qq; 2\varepsilon), L(Tq, Pp; 2\varepsilon)\}$$

where $\mathcal{F}(p, q)$ is defined by (4.2.1). Let $t = \dfrac{H_\lambda(Pp, Qq)}{h_\lambda}$.

Then

$$t \le \max\ \{d_\alpha(Tp, Tq), d_\alpha(Tp, Pp), d_\alpha(Tq, Qq),$$
$$1/2\ d_\alpha(Tp, Qq), 1/2\, d_\alpha(Tq, Pp)\}$$

for all $\alpha \in (0, 1)$,

and, in view of the definition of $\mathcal{F}$ (see (4.2.1)),

$$\min\ \{F(Tp, Tq; t), L(Tp, Pp; t), L(Tq, Qq; t), L(Tp, Qq; 2t),\ L(Tq, Pp; 2t)\} = 0,$$

that is $E(Pp, Pq; ht) < 0$.

This contradicts the nonnegativeness of E.

This completes the proof.

Theorem 4.4 — Let K be an arbitrary set, X a Hausdorff uniform space. Let P, Q be multivalued mappings from K to 2^X (the collection of nonempty closed subsets of X). If for every $d_\lambda \in \mathcal{P}$ there exists a constant $h_\lambda \in (0, 1)$ and mappings S, T : $K \to X$, such that for all x, $y \in K$, $\lambda \in I$,

$$H_\lambda(Px, Qy) \le h_\lambda \max\{d_\lambda(Sx, Ty), d_\lambda(Px, Sx),$$
$$d_\lambda(Qy, Ty), \frac{1}{2}\, d_\lambda(Px, Ty), \frac{1}{2}\, d_\lambda(Qy, Sx)\} \quad \text{... (4.4.1)}$$

and if $Q(K) \subset S(K), P(K) \subset T(K)$ and $S(K)$ (resp. $T(K)$) is a sequentially complete subspace of X then P and S (resp. Q and T) have a coincidence.

PROOF : The proof is analogous to the proof of Theorem 4.3.

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