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THE GLOBAL SET-DOMINATION NUMBER OF A GRAPH

E. SAMPATHKUMAR AND L. PUSHPA LATHA

Department of Mathematics, University of Mysore, Mysore 570 006

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Let G be a co-connected graph. A set $D \subset V$ is a 'set-dominating set' (sd-set) if for every set $S \subset V - D$, there exists a nonempty set $T \subset D$ such that the subgraph $\langle S \cup T \rangle$ is connected. Further, D is a global sd-set if D is an sd-set of both G and $\overline{G}$. The 'set-domination number' γ_S and the 'global set-domination number' γ_{Sg} of G are defined as expected.

Theorem 1 — For a tree of order p with e end vertices, $\gamma_{Sg} = p - e$.

Theorem 2 — If diam $G = 3$, then $\gamma_{Sg} \leq \gamma_S + 2$;
if diam $G = 4$, then $\gamma_{Sg} \leq \gamma_S + 1$;
if diam $G \geq 5$, then $\gamma_{Sg} = \gamma_S$.

Let $G = (V, E)$ be a graph. A set $D \subset V$ is a dominating set of G if every vertex not in D is adjacent to some vertex in D. Further, D is a global dominating set of G if D is a dominating set of both G and its complement $\overline{G}$. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The global domination number $\gamma_g(G)$ of G is defined similarly. The concept of global domination was first introduced by Sampathkumar[4], and was also studied by Rall[3]. Recently, the concept of set domination for a connected graph was introduced by the authors[5]. A set $D \subset V$ is an set-dominating set (sd-set) if for every set $S \subset V - D$, there exists a nonempty set $T \subset D$ such that the subgraph $\langle S \cup T \rangle$ induced by $S \cup T$ is connected. The set-domination number $\gamma_s = \gamma_s(G)$ of G is the minimum cardinality of an sd-set. Suppose G is a co-connected graph (i.e. both G and $\overline{G}$ are connected). The global set-domination number $\gamma_{sg} = \gamma_{sg}(G)$ of G is the minimum cardinality of an sd-set of both G and $\overline{G}$. The purpose of this paper is to initiate a study of γ_{sg}.

Henceforth, we consider only co-connected graphs G. For a vertex v in G, let $N(v) = \{u : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. Also $\overline{\gamma}_s = \gamma_s(\overline{G})$.

Since every global sd-set is a global dominating set, and $\gamma_g \geq 2$, we have

$$2 \leq \gamma_g \leq \gamma_{sg}. \qquad \text{... (1)}$$

We observe that for a path P_n on $n \geq 4$ vertices, $\gamma_{sg}(P_n) = n - 2$, and for a cycle C_n on $n \geq 6$ vertices, $\gamma_{sg}(C_n) = n - 3$, whereas $\gamma_{sg}(C_5) = 3$.

A γ_s-set is a minimum sd-set. Similarly, we define a γ_{sg}-set etc.

One can easily determine γ_{sg} for a tree.

Proposition 1 — In a tree T with p vertices and e end vertices, that is not a star, the set of non-end vertices forms a minimum global sd-set and $\gamma_{sg} = p - e$.

PROOF : It is known that the set D of all cut vertices of T forms a γ_s-set of T and $\gamma_s = p - e$ (see Sampathkumar and Pushpa Latha[5]). Clearly, the subgraph $\langle V(T) - D \rangle$ in $\bar{T}$ is complete. Since $T \neq K_{1,n}$, in $\bar{T}$, each vertex in $V(T) - D$ is adjacent to some vertex in D. This implies that D is an sd-set of $\bar{T}$ also, and $\gamma_{sg} = p - e$.

We now determine some elementary bounds for γ_{sg}.

Proposition 2 — Let G be a co-connected graph of order $p \geq 4$. Then,

$$2 \leq \gamma_{sg} \leq p - 2. \qquad \text{... (2)}$$

PROOF : Let u and v be adjacent vertices of degree at least two (such vertices clearly exist). Then $V - \{u, v\}$ is a global sd-set of G, so $\gamma_{sg} \leq p - 2$.

The bounds in (2) are sharp. The upper bound is attained by paths of length at least 3 and the 5-cycle. All graphs for which the lower bound is attained can be determined.

Proposition 3 — For a graph G of order p, $\gamma_{sg} = 2$ if and only if diam G = diam $\bar{G}$ = 3 and either G or $\bar{G}$ has a bridge which is not an end edge.

PROOF : Assume $\gamma_{sg} = 2$. Since diam $G \leq \gamma_s + 1$ (see Sampathkumar and Pushpa Latha[5]), we have diam $G \leq 3$ and diam $\bar{G} \leq 3$. Now, let $D = \{u, v\}$ be a γ_{sg}-set of G. Suppose u and v are adjacent in G. All vertices in $V(G) - D$ are adjacent to either u or v (but not to both). If all such vertices are adjacent to only u (or v) in G, then $\bar{G}$ is disconnected. Hence, some vertices of $V(G) - D$ are adjacent to u and some are adjacent to v. If $x \in N(u) - \{v\}$ and $y \in N(v) - \{u\}$, then x and y are not adjacent in G, for otherwise, $\{u, v\}$ will not be an sd-set in $\bar{G}$. Thus, uv is a bridge in G that is not an end edge, and $d(x, y) = 3$ = diam G. Also in $\bar{G}$, $d(u, v) = 3$ and hence, diam $\bar{G} = 3$.

Conversely, if G has a bridge uv that is not an end edge, and diam G = diam $\bar{G} = 3$, then every vertex in G is adjacent to u or to v and hence $\{u, v\}$ is a γ_s-set in G. Let $N_G(u)$ be the set of all neighbours of u in G. Then, $N_G(u) = N_{\bar{G}}[v]$. Since uv is a bridge in G, every vertex of $N_G(u) - \{v\}$ is adjacent to every vertex of $N_G(v) - \{u\}$ in $\bar{G}$. Hence $\{u, v\}$ is an sd-set of $\bar{G}$, and $\gamma_{sg} = 2$.

A connected dominating set of a graph G is a dominating set D of G such that the subgraph $\langle D \rangle$ is connected. The 'connected domination number' γ_c of G is the minimum cardinality of a connected dominating set of G (see Hedetniemi and Laskar[2] and Sampathkumar and Walikar[6]).

In the previous paper[5], it is shown that for a graph G with cut vertices, $\gamma_c = \gamma_s$. We now investigate graphs for which γ_s and γ_{sg} differ by at most one.

Proposition 4 — Let G be a graph with cut vertices. Then,

$$\gamma_{sg} \leq \gamma_s + 1 = \gamma_c + 1.$$

PROOF : We consider two cases.

Case 1 — There exists a γ_s-set D of G all of whose vertices belong to a single block B of G.

Consider a vertex $u \notin D$ such that u is in a block $B_1 \neq B$. Let $D' = D \cup \{u\}$. We now show that D' is an sd-set of G. Let $v, w \in V(\overline{G})$. If v, w belong to a single block $B_i \neq B_1$ of G, then they are both adjacent to u in $\overline{G}$. If v and w are in B_1, then in $\overline{G}$, both of them are adjacent to a vertex $u_1 \in D \cap (B - B_1)$ (note that $\gamma_s \geq 2$). If $v \in B_1$ and $w \notin B_1$, then in $\overline{G}$, v is adjacent to u_1 and w is adjacent to u. Further, the subgraph $\langle \{u, v, w, u_1\} \rangle$ is connected in G. This proves that D' is an sd-set of G and $\gamma_{sg} \leq |D'| = \gamma_s + 1$.

Case 2 — Case 1 is not true.

In this case, for every γ_s-set D of G, at least two vertices of D belong to different blocks of G. One can easily show that D is also an sd-set of $\overline{G}$. Hence $\gamma_{sg} = \gamma_s = \gamma_c$.

Corollary 4.1 — If G has two cut vertices that do not lie on a cycle, then $\gamma_c = \gamma_s = \gamma_{sg}$.

PROOF : This follows from Proposition 4, since it is known that there always exists a γ_s-set D of G containing all cut vertices (see Sampathkumar and Pushpa Latha[5]).

Let diam $G = k$. We show that when $k \geq 3$, γ and γ_g, as well as γ_s and γ_{sg} differ by at most two. In particular, we prove $\gamma_s = \gamma_{sg}$ when $k \geq 5$. Note that Rall[3] has shown that $\gamma = \gamma_g$ when $k \geq 5$.

Proposition 5 — Let G be a graph having diameter at least five, and let $D \subset V(G)$. Then D is a minimal sd-set of G if and only if D is a minimal global sd-set of G.

PROOF : Suppose D is a minimal sd-set of G. Let u and v be such that $d(u, v) \geq 5$. Then $D \cap N[u] \neq \phi$ and $D \cap N[v] \neq \phi$. Let $u_1 \in D \cap N[u]$ and $v_1 \in D \cap N[v]$. Since $d(u, v) \geq 5$, u_1 and v_1 are nonadjacent in G, and hence they are adjacent in $\overline{G}$. Also no vertex in $V(G) - \{u_1, v_1\}$ is adjacent to both u_1 and v_1 in G, since otherwise, $d(u, v) < 5$.

Now, in $\overline{G}$, each vertex is adjacent to u_1 or v_1 (or both) and hence $\{u_1, v_1\}$ is a connected dominating set of $\overline{G}$. Since every connected dominating set is an sd-set, $\{u_1, v_1\}$ is an sd-set of $\overline{G}$. This proves that D is an minimal global sd-set of G.

Conversely, if D is a minimal global sd-set of G, and is not a minimal sd-set of G, then there exists $x \in D$ such that $D - \{x\}$ is also an sd-set of G. As before, if $u_1 \in (D - \{x\}) \cap N[u]$ and $v_1 \in (D - \{x\}) \cap N[v]$, then $\{u_1, v_1\}$ is an sd-set of $\overline{G}$, and hence $D - \{x\}$ is a global sd-set of G, a contradiction. Hence, D is also a minimal sd-set of G.

Corollary 5.1 — If diam $G \geq 5$, then $\gamma_s = \gamma_{sg}$.

Proposition 6 — Let G be a co-connected graph.

1. When diam $G = 4$, (a) $\gamma_g \leq \gamma + 1$, and (b) $\gamma_{sg} \leq \gamma_s + 1$.
2. When diam $G = 3$, (a) $\gamma_g \leq \gamma + 2$, and (b) $\gamma_{sg} \leq \gamma_s + 2$.

PROOF : We prove only 1(b) and 2(b) since the proofs of 1(a) and 2(a) are so similar.

1(b) : Let u and v be such that $d(u, v) = 4$. Suppose A is a γ_s-set of G. Then $A \cap N[u] \neq \phi$. Let $u_1 \in A \cap N[u]$. Since $d(u, v) = 4$, no vertex in G is adjacent to both u_1 and v. Consider the set $\{u_1, v\}$. In $\bar{G}$, u_1 and v are adjacent, and each vertex in $V(G) - \{u_1, v\}$ is adjacent to at least one of them. Hence $\{u_1, v\}$ is an sd-set of $\bar{G}$ and $A \cup \{v\}$ is a global sd-set of G. Thus $\gamma_{sg}(G) \leq |A \cup \{v\}| \leq \gamma_s(G) + 1$.

2(b) : Let $u, v \in V(G)$ with $d(u, v) = 3$, and let A be a γ_s-set of G. Since $d(u, v) = 3$, no vertex of G is adjacent to both u and v. In $\bar{G}$, u and v are adjacent, and every vertex in $V(G) - \{u, v\}$ is adjacent to at least one of them. Hence $\{u, v\}$ is an sd-set of $\bar{G}$. This implies that $A \cup \{u, v\}$ is a global sd-set of G, and 2(b) follows.

The bounds for γ_g and γ_{sg} in 1 and 2 are sharp. For example, for the graph G of diameter 4 in Fig. 1, $\{u, v\}$ is a γ-set whereas $\{u, v, y\}$ is a γ_g-set. Thus, $\gamma = 2$ and $\gamma_g = 3$. Also $\{u, v, w\}$ is a γ_s-set of G and $\{u, v, w, y\}$ is a γ_{sg}-set. Hence $\gamma_s = 3$, $\gamma_{sg} = 4$. For the graph H of diameter 3 in Fig. 1, $\{a, b\}$ forms a γ-set as well as a γ_s-set, and the γ_g-set $\{a, b, c, d\}$ is also a γ_{sg}-set. Thus, for H, $\gamma = \gamma_s = 2$, and $\gamma_g = \gamma_{sg} = 4$.

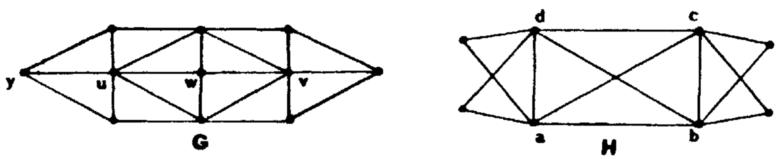

FIG. 1.

For any given positive integer n, there exist graphs of diameter two such that $\gamma_g - \gamma = n$, and $\gamma_{sg} - \gamma_s = n$. For example, if $G = \bar{C}_{n+5}$, then diam $G = 2$, $\gamma_s = 2$ and $\gamma_{sg} = n + 2$. Similarly, for the graph $H = \bar{C}_{3n+6}$, of diameter 2, we have $\gamma = 2$, and $\gamma_g = n + 2$.

CONCLUDING REMARKS

Given a graph G, one can ask the question : What are the graphical parameters concerning both G and $\bar{G}$? To the best of our knowledge, the global domination number was the first such parameter, introduced by Sampathkumar[4]. This motivates one to introduce other similar parameters (as we have done here) and study their relationship with the parameters defined for G. Recently, another graphical parameter of this type was studied by Dunbar *et al.*[1].

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