Indian J. pure appl. Math., **25**(10): 1059-1065, October 1994

ON ITERATED LINE GRAPHS WITH CROSSING NUMBER ONE

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(*Received* 9 *February* 1993; *after revision* 10 *May* 1994; *accepted* 24 *May* 1994)

In this paper, we obtain a necessary and sufficient condition for an *n*th line graph $L^n(G)$ for $n \geq 2$, to have crossing number 1.

1. PRELIMINARIES

For a graph G, let $V(G)$, $E(G)$ and $L(G)$ denote as usual the vertex set, the edge set and the line graph of G and for a vertex v of G, let $d(v)$ denote its degree. The '*n*th iterated line graph' (in brief, '*n*th line graph') is defined as follows :

$$L^0(G) = G,$$

and

$$L^n(G) = L(L^{n-1}(G)) \text{ for } n \geq 1.$$

The 'crossing number' $cr(G)$ of a graph G is the minimum number of pairwise intersections of its edges when G is drawn in the plane. Obviously, $cr(G) = 0$ if and only if G is planar.

Sedlácek[2] has shown that the line graph of a graph G has crossing number 0 if and only if G is planar, the degree of each vertex is at most 4, and all vertices of degree 4 are cutvertices. Kulli *et al.*[1] characterized line graphs with crossing number 1. The purpose of this paper is to characterize all *n*th line graphs with crossing number 1 for $n \geq 2$.

The following theorems of Kulli *et al.*[1] will be applied in proving our results.

Theorem 1 — The crossing number of the line graph of a non-planar graph is at least 3.

Theorem 2 — The line graph of a graph G has crossing number 1 if and only if G is planar and one of the following conditions holds :

* Research supported by CSIR, New Delhi.

(a) $\Delta(G) = 4$, and G has exactly one non-cutvertex of degree 4.

(b) $\Delta(G) = 5$, every vertex of degree 4 is a cutvertex, there is exactly one vertex of degree 5 and it has at most three edges in any block of G.

2. Results

We now establish a necessary and sufficient condition for a second line graph to have crossing number 1.

First, we introduce the notion of the total degree of an edge. The 'total degree' of an edge uv in a graph G, denoted $d'(uv)$, is defined as $d(u) + d(v)$. We denote by $\Delta'(G)$ the maximum total degree of edges of G.

Theorem 3 — For any graph G, $cr(L^2(G)) = 1$ if and only if G is planar, $\Delta(G) \leq 4$, and every edge incident with a vertex of degree 4 is a bridge. Moreover, one of the following conditions holds :

(i) $\Delta'(G) = 6$ and there is exactly one edge of total degree 6 that is not a bridge.

(ii) $\Delta'(G) = 7$, there is exactly one edge of total degree 7 and every edge of total degree 6 is a bridge.

Proof : Suppose $cr(L^2(G)) = 1$. Then by Theorem 1, $L(G)$ is planar. So, G is planar and $\Delta(G) \leq 4$ from Sedlácek's result. Let v be a vertex of degree 4 in G. Denote the edges at v by e_1, e_2, e_3 and e_4. If some e_i, say for $i = 1$, is not bridge, then e_1 and one of the edges e_j for $j = 2$, lie on a cycle of G. In $L(G)$, there is a cycle containing the vertices e_1 and e_2 but not e_3 and e_4, and moreover $\langle\{e_1, e_2, e_3, e_4\}\rangle$ is isomorphic to K_4. This implies that $L(G)$ has two vertices e_1 and e_2, each of degree 4 that are not cutvertices. By Theorem 2, $cr(L^2(G)) > 1$, a contradiction.

Since $\Delta(G) \leq 4$, $d'(e) \leq 8$ for every edge e of G. Assume that $d'(e) = 8$ for some edge e of G. Then in $L(G)$, the vertex e has degree 6. Therefore by Theorem 2, $cr(L^2(G)) > 1$, contrary to our assumption. Next, suppose that $d'(e) \leq 5$ for each edge e of G. Then $L(G)$ contains a vertex of degree at most 3, and hence $\Delta(L(G)) \leq 3$. By Sedlácek's theorem, $L^2(G)$ has no crossings in an optimal drawing, a contradiction. Thus, there is an edge in G with total degree either 6 or 7.

Case 1 — Suppose $\Delta'(G) = 6$. Then G has at least one edge with total degree 6. This implies that $\Delta(L(G)) = 4$. Now, we check that G contains at least one non-bridge of total degree 6. If this is not the case, then each bridge e with total degree 6 corresponds to a cutvertex e of degree 4 in $L(G)$. Consequently, all vertices of degree 4 in $L(G)$ are cutvertices. Thus, $cr(L^2(G)) = 0$, contrary to our assumption. Finally, suppose that G has two edges of total degree 6 that are not bridges. Then each such edge of G corresponds to a non-cutvertex of degree 4 in $L(G)$. So, $L(G)$ has two non-cutvertices of degree 4. From Theorem 2, $cr(L^2(G)) > 1$, a contradiction.

Case 2 — Suppose $\Delta'(G) = 7$. Then G has at least one edge with total degree 7. Thus, $\Delta(L(G)) = 5$. If G contains more than one edge of total degree 7, then $L(G)$ has more than one vertex of degree 5. By Theorem 2, $cr(L^2(G)) > 1$, a contradiction.

Finally, if there were also a non-bridge e in G with $d'(e) = 6$ then (as in Case 1) $L(G)$ would contain a non-cutvertex of degree 4, and hence it would contribute at least one crossing in an optimal drawing of $L^2(G)$. But then $L^2(G-e)$ is non-planar. Therefore, $L^2(G)$ would contain at least two crossings in this case, a contradiction.

Conversely, suppose that a planar graph G satisfies the hypothesis of the theorem. Clearly, $L(G)$ is planar and the degree of a vertex e in $L(G)$ is $d'(e) - 2 \le 5$. Thus, $\Delta(L(G)) \le 5$.

Now, assume (i) holds. Then there is exactly one vertex of degree 4 in $L(G)$ that is not a cutvertex. From Theorem 2, $cr(L^2(G)) = 1$.

Finally, assume (ii) holds. Because G has exactly one edge e of total degree 7, $L(G)$ contains exactly one cutvertex e of degree 5 which lies on exactly two cliques K_3 and K_4. Moreover, every edge of total degree 6 in G is a bridge, then each such bridge of G corresponds to a cutvertex of degree 4 in $L(G)$. Again by Theorem 2, $cr(L^2(G)) = 1$. ■

Next, we introduce definitions for our later use.

Definition 1 — The 'neighbourhood degree' (in brief, N-degree) of a vertex u in a graph G is the sum of the degrees of the vertices which are adjacent to u.

Definition 2 — A graph G^+ is called 'skew homeomorphic to a graph G with respect to an edge x' (in brief, 'x-skew homeomorphic to G') if G^+ is obtained from G by a finite sequence of subdivisions of any edges of G except the edge x.

In the following theorem, we give a necessary and sufficient condition for graphs whose third line graphs have crossing number 1.

Theorem 4 — For any graph G, $cr(L^3(G)) = 1$ if and only if the following conditions hold :

(a) G is planar,

(b) $\Delta(G) = 3$, and

(c) G has exactly one vertex of degree 3 with N-degree 5, and all other vertices of degree 3 have N-degree at most 4.

PROOF : Suppose $cr(L^3(G)) = 1$. Then $L(G)$ satisfies the conditions of Theorem 3; in particular, because $L(G)$ is planar, G is planar and $\Delta(G) \le 4$.

Suppose $\Delta(G) \le 2$. Then each component of G is either a cycle or a path. So, each component of $L^n(G)$ for $n \ge 1$ is either a cycle or a path. This implies that $cr(L^3(G)) = 0$, a contradiction. Suppose $\Delta(G) = 4$. Then $L(G)$ contains a subgraph isomorphic to K_4. Since every edge of K_4 is a non-bridge with total degree 6, there are at least six non-bridges in $L(G)$. From Theorem 3, $cr(L^3(G)) > 1$, a contradiction. So, the above discussion yields $\Delta(G) = 3$ as the only possibility for G.

Next, suppose that each vertex of degree 3 has N-degree ≤ 4. Then it is easily seen that $cr(L^3(G)) = 0$, a contradiction. This implies that G has at least one vertex of degree 3 with N-degree ≥ 5. Assume that G has a vertex with N-degree ≥ 6. Then G has a subgraph $H = H_1$ or it is homeomorphic to H_2 as shown in Fig. 1.

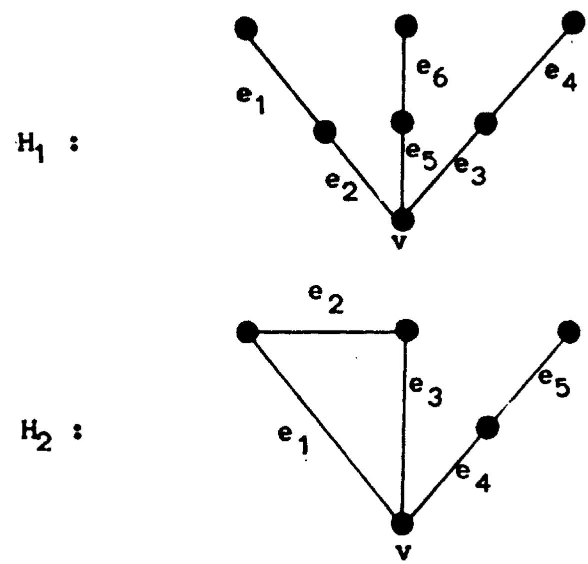

FIG. 1.

In either case, $L(H)$ and hence $L(G)$, contains three mutually adjacent non-bridges, each of total degree 6. This implies that $cr(L^3(G)) > 1$, by Theorem 3. Therefore, each vertex of degree 3 in G has N-degree 5.

All that remains is to show that there is exactly one vertex of degree 3 and N-degree 5. Suppose this is not the case. Let u and v be two vertices of degree 3 in G, each has N-degree 5. Then G has a subgraph A which is one of the following types as shown in Fig. 2.

Type I : A is isomorphic to either A_1 or $2A_2$.

Type II : A is x-skew homeomorphic to either $2A_3$ or $A_2 \cup A_3$.

Assume that A is of Type I. If $A = A_1$, then $L(A) = K_1 + 2K_2$ and has four non-bridges $e_1 e_i$ for $i = 2, 3, 4$ and 5 such that $d'(e_1 e_i) = 6$. Since $L(A)$ is a subgraph of $L(G)$, as before, $L^3(G)$ has more than one crossing, a contradiction. If $A = 2A_2$, then $L(A)$ has two components, each isomorphic to $L(A_2)$. It is easy to check that $L(A_2)$ in $L(G)$ contains a non-bridge $e_1 e_2$ such that $d'(e_1 e_2) = 6$. Thus, $L(G)$ has at least two non-bridges, each with total degree 6. As before, $cr(L^3(G)) > 1$, a contradiction.

Suppose A is of Type II. If A is x-skew homeomorphic to $2A_3$, then $L(A)$ has two components, each isomorphic to $L(A_3) = K_2 + \overline{K}_2$. It is easy to see that each $L(A_3)$ in $L(G)$ contains a non-bridge $e_1' e_2'$ such that $d'(e_1' e_2') = 6$. Thus, $L(G)$ has at least two non-bridges, each of total degree 6. As before, $L^3(G)$ has more than one crossing, a contradiction. Finally, if A is x-skew homeomorphic to $A_2 \cup A_3$, then

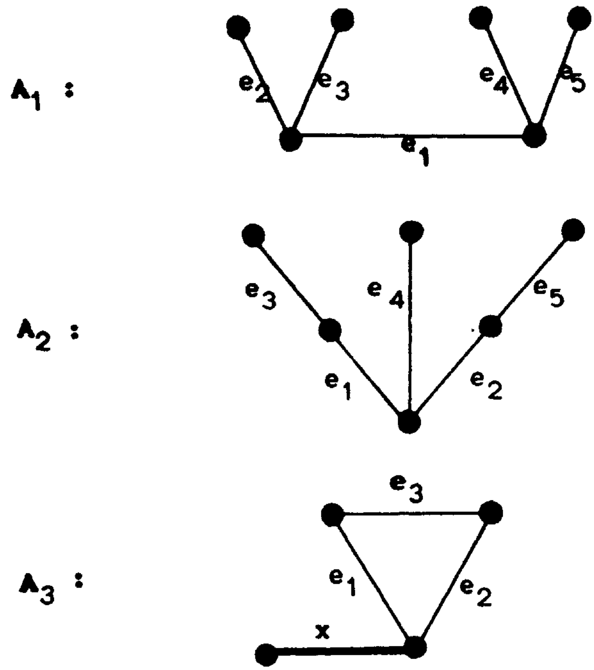

FIG. 2.

arguing as above, we see that $L(A)$ has two components $L(A_2)$ and $L(A_3)$, each contains a non-bridge with total degree 6. Hence $L^3(G)$ has at least two crossings. This again leads to a contradiction.

Conversely, suppose G is a planar graph satisfying the conditions of the theorem. Then $L(G)$ satisfies the conditions of Theorem 3, and hence $cr(L^3(G)) = 1$. ■

Next, we obtain a characterization of graphs whose fourth line graphs have crossing number 1.

Theorem 5 — For any graph G, $cr(L^4(G)) = 1$ if and only if the following conditions hold :

(a) G is planar,

(b) $\Delta(G) = 3$, and

(c) G has exactly one vertex of degree 3 with N-degree 4, and all other vertices of degree 3 have N-degree at most 3.

PROOF : Suppose $cr(L^4(G)) = 1$. Then $L(G)$ satisfies the conditions of Theorem 4; in particular, by producing the same argument as in the proof of Theorém 4, we obtain that G is planar and $\Delta(G) = 3$.

Next, assume that each vertex of degree 3 in G has N-degree 3. Then the component of G containing such a vertex is isomorphic to $K_{1,3}$. Obviously, then

$L^4(G)$ has crossing number 0, a contradiction. Hence G has at least one vertex v of degree 3 and N-degree ≥ 4.

Suppose the N-degree of v in G is ≥ 5. Then G has a subgraph $B = A_2$ or it is homeomorphic to A_3 as shown in Fig. 2. If $B = A_2$, then $L(B)$ has two vertices of degree 3 and N-degree 6. But $L(B)$ is in $L(G)$. From Theorem 4, $L^4(G)$ has more than one crossing, a contradiction. If B is homeomorphic to A_3, then $L(B) = K_2 + \bar{K}_2$ and has two vertices e_1' and e_2', each of degree 3 and N-degree 7. As above, $cr(L^4(G)) > 1$, a contradiction. This shows that each vertex of degree 3 in G has N-degree 4.

Finally, all that remains is to show that there is exactly one vertex of degree 3 and N-degree 4. Suppose not; that is, G has two vertices of degree 3 and N-degree 4. Then G contains a subgraph H isomorphic to $2D$ as shown in Fig. 3. Consequently, $L(H)$ has two components, each isomorphic to $K_1 + (K_1 \cup K_2)$. Moreover, it has the vertex e_3 of degree 3 in $L(G)$ and has N-degree 5. Thus, $L(G)$ has at least two vertices of degree 3 and N-degree 5. As before, $cr(L^4(G)) > 1$, a contradiction.

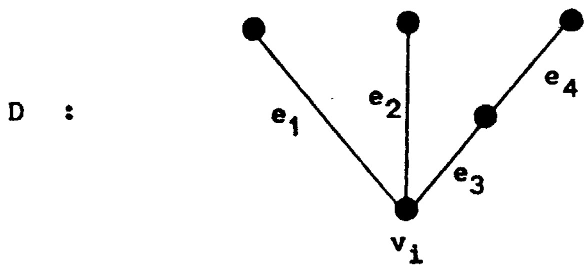

FIG. 3.

Conversely, suppose G is a planar graph satisfying the given conditions. Then $L(G)$ satisfies the conditions of Theorem 4, and therefore $cr(L^4(G)) = 1$. ■

Theorem 6 — For $n \geq 5$, there is no graph whose nth line graph has crossing number 1.

PROOF : Suppose G is non-planar. Then, in view of Theorem 1, $cr(L^n(G)) \geq 3$ for all $n \geq 1$.

Suppose G is planar. Then we consider three cases depending on the magnitude of $\Delta(G)$:

Case 1 — $\Delta(G) \leq 2$. Then every component of G and hence of $L^n(G)$ for $n \geq 1$ is either a cycle or a path. Hence $cr(L^n(G)) = 0$.

Case 2 — $\Delta(G) = 3$. Then G contains a vertex v of degree 3. If the N-degree of v is 3, then the component of G containing v is isomorphic to $K_{1,3}$ and therefore, $cr(L^n(G)) \neq 1$. If the N-degree of v is ≥ 4, then from Theorem 5, $L^4(G)$ is non-planar. By Theorem 1, $L^n(G)$ has at least three crossings for $n \geq 5$.

Case 3 — $\Delta(G) \geq 4$. Then from Theorem 4, $cr(L^n(G)) > 1$ for $n \geq 3$. ■

ACKNOWLEDGEMENT

The authors are grateful to the referees for their valuable suggestions and helpful comments in improving the earlier version of this paper.

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