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ON SOME SEQUENCE SPACES

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In this paper we introduce the generalization of the lacunary strongly convergent sequences and give the relation between lacunary strong convergence and lacunary strong convergence with respect to a modulus.

The spaces of lacunary strong convergence have been introduced by Freedman *et al.*[4]. A sequence of positive integers $\theta = (k_r)$ is called 'lacunary' if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$. The space of lacunary strongly convergent sequences N_θ was defined as follows :

$$N_\theta = \{x = (x_i) : \lim_{r\to\infty} h_r^{-1} \sum_{i\in I_r} |x_i - s| = 0 \text{ for some } s\}.$$

Let $\|x\|_\theta = \sup_r (h_r^{-1} \sum_{i\in I_r} |x_i|)$, whenever $x \in N_\theta$. Then $(N_\theta, \|\cdot\|_\theta)$ is a BK-space. N_θ^0 denotes the subset of all sequences which are lacunary strongly. convergent to zero. $(N_\theta^0, \|\cdot\|_\theta)$ is also a BK-space. In the special case $\theta = (2^r)$, we have $N_\theta = w$, where w is the space of strongly summable sequences.

The famous space $\hat{c}$ of all almost convergent sequences was defined by Lorentz[7] and the space of strongly almost convergent sequences $[\hat{c}]$ was defined by Maddox[9] as follows :

$$[\hat{c}] = \{x = (x_i) : \lim_{n\to\infty} n^{-1} \sum_{i=p+1}^{p+n} |x_i - s| = 0 \text{ uniformly in } p, \text{ for some } s\}.$$

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The space of lacunary almost convergent sequences $\hat{c}_\theta$ and the space of lacunary strongly almost convergent sequences $[\hat{c}_\theta]$ have been studied by Das and Patel[2].

The notion of a modulus function was introduced by Nakano[11]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x+y) \le f(x) + f(y)$ for $x, y \ge 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows that f must be continuous on $[0, \infty)$. A modulus may be bounded or unbounded. Maddox[9, 10] and Ruckle[14] used a modulus f to construct sequence spaces. In an earlier paper[12], the spaces $[\hat{c}_0]$ and $[\hat{c}]$ were extended to $[\hat{c}_0(f)]$ and $[\hat{c}(f)]$.

We now introduce the generalizations of the lacunary strongly convergent sequences and give the relation between lacunary strong convergence and lacunary strong convergence with respect to a modulus.

Definition — Let f be a modulus. We define the spaces

$$N_\theta(f) = \{x = (x_i) : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) = 0, \text{ for some } s\},$$

$$N_\theta^0(f) = \{x = (x_i), : \lim_{r \to \infty} h_r^{-1} \sum_{i \in I_r} f(|x_i|) = 0\}.$$

Note that if we put $f(x) = x$, then $N_\theta(f) = N_\theta$ and $N_\theta^0(f) = N_\theta^0$. Both $N_\theta(f)$ and $N_\theta^0(f)$ are linear spaces. We consider only $N_\theta(f)$. Suppose that $x_i \to s$ and $y_i \to s'$ in $N_\theta(f)$ and that α, γ are in C, the complex numbers. Then there exist integers T_α and M_γ such that $|\alpha| \le T_\alpha$ and $|\gamma| \le M_\gamma$. We therefore have

$$h_r^{-1} \sum_{i \in I_r} f(|\alpha x_i + \gamma y_i - (\alpha s + \gamma s')|)$$

$$\le T_\alpha h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) + M_\gamma h_r^{-1} \sum_{i \in I_r} f(|x_i - s'|).$$

This implies that $\alpha x + \gamma y \to \alpha s + \gamma s'$ in $N_\theta(f)$.

Proposition 1 — Let f be a modulus and let $0 < \delta < 1$. Then for each $x \ge \delta$ we have $f(x) \le 2f(1)\delta^{-1}x$.

PROOF : $f(x) \le f(1 + [x/\delta]) \le f(1) + f([x/\delta]) \le f(1) + [x/\delta]f(1)$

$$= f(1)(1 + [x/\delta]) \le f(1)(1 + x/\delta) \le 2f(1)x/\delta,$$

where $[x/\delta]$ denotes the integer part of x/δ.

Proposition 2[10] — Let f be any modulus. Then $\lim_{t\to\infty} f(t)/t = \beta$ exists.

Theorem 3 — Let f be any modulus. If $\lim_{t\to\infty} f(t)/t = \beta > 0$, then $N_\theta(f) = N_\theta$.

PROOF : If $x \in N_\theta$, then

$$A_r = h_r^{-1} \sum_{i \in I_r} |x_i - s| \to 0 \text{ as } r \to \infty, \text{ for some } s.$$

Let $\varepsilon > 0$ be given. We choose $0 < \delta < 1$ such that $f(u) < \varepsilon$ for every u with $0 \le u \le \delta$. We can write

$$h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) = h_r^{-1} \sum_{\substack{i \in I_r \\ |x_i - s| \le \delta}} f(|x_i - s|) + h_r^{-1} \sum_{\substack{i \in I_r \\ |x_i - s| > \delta}} f(|x_i - s|)$$

$$\le h_r^{-1}(h_r \varepsilon) + 2f(1)\,\delta^{-1} A_r,$$

by Proposition 1 as $r \to \infty$. Therefore $x \in N_\theta(f)$.

Note that in this part of the proof we do not need $\beta > 0$. Now let $\beta > 0$ and let $x \in N_\theta(f)$. Since $\beta > 0$, we have $f(t) \ge \beta t$ for all $t \ge 0$. It follows that $x \in N_\theta(f)$ implies that $x \in N_\theta$.

As an example to show that $N_\theta(f) \ne N_\theta$ when $\beta = 0$, consider the modulus $f(x) = \ln(1 + x)$. In this case $\beta = 0$. Define x_i to be h_r at the $(k_{r-1} + 1)$-th term in I_r for every $r \ge 1$ and $x_i = 0$ otherwise. Note that x is not bounded. Then we have

$$h_r^{-1} \sum_{i \in I_r} f(|x_i|) = h_r^{-1} \ln(1 + h_r) \to 0 \text{ as } r \to \infty$$

and so $x \in N_\theta(f)$. But

$$h_r^{-1} \sum_{i \in I_r} |x_i| = h_r^{-1} h_r \to 1 \text{ as } r \to \infty$$

and so $x \notin N_\theta$.

Corollary 4 — $[\hat{c}(f)] \subset N_\theta(f)$ for every lacunary sequence θ.

For instance, to give a sequence in $N_\theta^0(f)$ and not in

$$[\hat{c}_0(f)] = \{x = (x_i) : \lim_{n \to \infty} n^{-1} \sum_{i=1}^{n} f(|x_{i+p}|) = 0 \text{ uniformly in } p\}.$$

Define $x = (x_i)$ by $x_i = 1$ if $k_{r-1} < i \le k_{r-1} + [\sqrt{h_r}]$ for some r and $x_i = 0$ otherwise. Then there are arbitrarily long strings of consecutive 0's in the coordinates of x, as well as arbitrarily long strings of consecutive 1's, from which it follows that $x \notin [\hat{c}_0(f)]$. However, $x \in N_\theta^0(f)$ since

$$h_r^{-1} \sum_{i \in I_r} f(|x_i|) = h_r^{-1} [\sqrt{h_r}] f(1) \to 0 \text{ as } r \to \infty.$$

Theorem 5 — Let f be a bounded modulus function. Then a necessary and sufficient condition for $\sum_{i=1}^{\infty} a_i x_i$ to be convergent whenever $x = (x_i) \in N_\theta(f)$ is that $a = (a_i) \in \phi = \{x : x \text{ is finitely nonzero}\}$.

PROOF : Since f is bounded, there exists an integer H such that $f(x) < H$ for all $x \geq 0$. Since $a \in \phi$, $\sum_{i=1}^{\infty} a_i x_i$ reduces to a finite sum and so the sufficiency is trivial.

For the necessity, suppose that $a \notin \phi$. Then there is a sequence of positive integers $k_1 < k_2 < ....$ such that $|a_{k_r}| > 0$ for $r \geq 1$. We define $x_i = a_i^{-1}$ for $i = k_1, k_2, ...$ and $x_i = 0$ otherwise. We have

$$h_r^{-1} \sum_{i \in I_r} f(|x_i|) \leq H h_r^{-1}$$

and so $x \in N_\theta(f)$, but $\sum_{i=1}^{\infty} a_i x_i$ diverges. The necessity now follows.

The notion of statistical convergence was given in earlier works[1, 3, 5, 13]. Recently, Fridy and Orhan[6] introduced the concept of lacunary statistical convergence :

Let θ be a lacunary sequence. Then a sequence $x = (x_i)$ is said to be lacunary statistically convergent to a number s if for every $\varepsilon > 0$,

$$\lim_{r \to \infty} h_r^{-1} |K_\theta(\varepsilon)| = 0,$$

where $K_\theta(\varepsilon) = \{i \in I_r : |x_i - s| \geq \varepsilon\}$ and $|K_\theta(\varepsilon)|$ denotes the number of elements in $K_\theta(\varepsilon)$. The set of all lacunary statistical convergent sequences is denoted by S_θ.

We now establish an inclusion relation between $N_\theta(f)$ and S_θ.

Theorem 6 — Let f be any modulus. Then $N_\theta(f) \subset S_\theta$.

PROOF : Suppose that $x \in N_\theta(f)$ and $\varepsilon > 0$. Then we have

$$h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) \geq h_r^{-1} \sum_{\substack{i \in I_r \\ |x_i - s| \geq \varepsilon}} f(|x_i - s|) > h_r^{-1} f(\varepsilon) |K_\theta(\varepsilon)|,$$

from which it follows that $x \in S_\theta$.

Theorem 7 — $S_\theta = N_\theta(f)$ if and only if f is bounded.

PROOF : Suppose that f is bounded and that $x \in S_\theta$. Since f is bounded, there exists a constant T such that $f(x) \le T$ for all $x \ge 0$. Splitting the sum $h_r^{-1} \sum_{i \in I_r} f(|x_i - s|)$ into two sums taken over $K_\theta(\varepsilon) = \{i \in I_r : |x_i - s| \ge \varepsilon\}$ and $\{i \in I_r : |x_i - s| < \varepsilon\}$, we see that

$$h_r^{-1} \sum_{i \in I_r} f(|x_i - s|) \le h_r^{-1} T |K_\theta(\varepsilon)| + f(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, the result follows.

Conversely, suppose that f is unbounded so that there exists a positive sequence $0 < t_1 < t_2 < \ldots < t_i < \ldots$ such that $f(t_i) \ge h_i$. Define the sequence $x = (x_i)$ by putting $x_{k_i} = t_i$ for $i = 1, 2, \ldots$ and $x_i = 0$ otherwise. We then have $x \in S_\theta$, but $x \notin N_\theta(f)$.

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