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TRIPLE GAUSSIAN HYPERGEOMETRIC FUNCTIONS I

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Eight solution types of the second order associated with the partial differential system of the triple hypergeometric function F_G are deduced. Hence, the systems of partial differential equations of each of these types are equivalent. In addition, the functions F_T and $F_D^{(3)}$ may be represented as special cases of certain of the above types and in themselves yield a further 4 and 2 distinct types respectively.

1. INTRODUCTION

About ten years ago, Srivastava and Karlsson[1] presented a comprehensive list of all of the triple hypergeometric functions of the second order which reduce to forms of the Gauss hypergeometric function ${}_2F_1$ when any two of the variables are suppressed. While a list of 14 such functions of two variables may be compiled, the list of functions of this type of three variables consists of no less than 205 distinct cases.

It has been noted that the systems of partial differential equations of the double Gaussian hypergeometric functions are in some cases mutually equivalent. For example, the systems associated with the functions F_1 and G_2 are equivalent as also are the systems associated with F_2, F_3 and H_2. See also Tables 1 and 2 of Kalnins *et al.* [2]. A detailed study of such equivalences has not been undertaken where the triple Gaussian hypergeometric functions are concerned. This is an extensive undertaking, and the present study is only applied to the triple series F_G and F_T. Lauricella[3] and Saran[4] first introduced these series which are defined as follows :

$$F_G\,(a, a, a, b_1, b_2, b_3; c_1, c_2, c_2; x, y, z)$$

$$= \sum \frac{(a, m+n+p)\,(b_1, m)\,(b_2, n)\,(b_3, p)\,x^m y^n z^p}{(c_1, m)\,(c_2, n+p)\,m!\,n!\,p!} \qquad \ldots (1.1)$$

and $$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c, c, c; x, y, z)$$

$$= \sum \frac{(a_1, m)(a_2, n+p)(b_1, m+p)(b_2, n)\, x^m y^n z^p}{(c, m+n+p)\, m!\, n!\, p!} \qquad \ldots (1.2)$$

and where throughout this paper, the indices of summation run over all the non-negative integers. In Srivastava and Karlsson's list, (1.1) and (1.2) are the same as 18a and 13a respectively.

2. A Pochhammer Integral for F_G

Saran[5] gave the following integral representation of the function F_G :

$$F_G(a+a'-1, a+a'-1, a+a'-1, b_1, b_2, b_3; c, d, d; x, y, z)$$

$$= \frac{\Gamma(a)\,\Gamma(a')\,\Gamma(2-a-a')}{(2\pi i)^2} \int (-t)^{-a}(t-1)^{-a'}\, {}_2F_1(a, b_1; c; x/t)$$

$$\times F_1(a', b_2, b_3; d; y/(1-t), z/(1-t))\, dt, \qquad \ldots (2.1)$$

where throughout this paper the contour of integration is a Pochhammer double-loop slung around the points 0 and 1 in the t-plane and where the hypergeometric functions of the integrand are given by

$${}_2F_1(a, b_1; c; x) = \sum \frac{(a, m)(b_1, m)}{(c, m)\, m!}\, x^m \qquad \ldots (2.2)$$

and $$F_1(a', b_2, b_3; d; y, z) = \sum \frac{(a', n+p)(b_2, n)(b_3, p)\, y^n z^p}{(d, n+p)\, n!\, p!} \qquad \ldots (2.3)$$

The Pochhammer symbol (a, m) denotes the product

$$a(a+1)(a+2) \ldots (a+m-1) = \Gamma(a+m)/\Gamma(a);\ (a, 0) = 1. \qquad \ldots (2.4)$$

3. Second-order Solution Types of the System F_G

Olsson[8] investigated in detail the partial differential system associated with the Appell function F_2, and observed that since the functions F_3 and H_2 are the only second order functions included among its solutions, then the partial differential systems associated with all three functions are equivalent. His approach was by the transformation of an integral formula for F_2. A rather similar method is applied to tackle the question of equivalent systems for the functions F_G and F_T.

In this integral on the right of (2.1), replace the hypergeometric functions of the integrand in turn by other solutions of the respective associated differential equations, when the result is also a solution of the differential system associated with F_G. Any such solutions which are not triple hypergeometric functions of the second order are discarded since the present interest is restricted to members of Srivastava and Karlsson's list.

Thus, we may replace ${}_2F_1(a, b_1; c; x/t)$ by

$$(x/t)^{-b_1}\,{}_2F_1(b_1, b_1 - c + 1; b_1 - a + 1; t/x), \qquad \ldots (3.1)$$

and we consider the solution

$$x^{-b_1}\int (-t)^{b_1-a}(t-1)^{-a'}\,{}_2F_1(b_1, b_1 - c + 1; b_1 - a + 1; t/x) \times F_1(a', b_2, b_3; d; y/(1-t), z/(1-t))\,dt. \qquad \ldots (3.2)$$

Integration term-by-term gives the series (apart from a constant multiplier)

$$x^{-b_1}\sum \frac{(b_1, m)(b_1 - c + 1, m)(b_2, n)(b_3, p)(a + a' - b_1 - 1, n + p - m)(-x)^{-m}\,y^n\,z^p}{(d, n + p)m!\,n!\,p!}. \qquad \ldots (3.3)$$

By comparison with Srivastava and Karlsson's list, it will be seen that (3.3) can be written as

$$x^{-b_1}\,\Phi_{5b}(b_2, b_3, b_1, b_1 - c + 1, a + a' - b_1 - 1; d; y, z, -1/x). \qquad \ldots (3.4)$$

It thus follows that the partial differential system associated with the functions F_G and Φ_{5b} are equivalent. The function Φ_{5b} is the same as $D^{1,3}_{(3)}$, a special case of which was introduced by Exton[9] in connection with the partial differential system associated with the Lauricella function F_D in three variables.

Now replace the Appell function $F_1(a', b_2, b_3; d; y/(1-t), z/(1-t))$ in (2.1) by

$$(y/(1-t))^{-b_2}(z/(1-t))^{-b_3}F_1(1 + b_2 + b_3 - d, b_2, b_3;\ 1 + b_2; + b_3 - a'; (1-t)/y, (1-t)/z) \qquad \ldots (3.5)$$

of Olsson's[7] list of solutions of the partial differential system associated with the function F_1. This gives the solution

$$y^{-b_2}z^{-b_3}\sum \frac{(b_1, m)(1 + b_2 + b_3 - d, n + p)(b_2, n)(b_3, p)(a + a' - 1 - b_2 - b_3, m - n - p)}{(c, m)} \times (x)^m(-1/y)^n(-1/z)^p, \qquad \ldots (3.6)$$

or $$y^{-b_2}z^{-b_3}\,\Phi_{5d}(1 + b_2 + b_3 - d, b_2, b_3, b_1, a + a' - 1 - b_2 - b_3; c; -1/y, -1/z, x) \qquad \ldots (3.7)$$

which is the same as $D^{2,3}_{(3)}$ in Exton[9].

If the two previous transformations are combined, we have the integral

$$x^{-b_1}y^{-b_2}z^{-b_3}\int (-t)^{b_1-a}(t-1)^{b_2+b_3-a'}\,{}_2F_1(b_1, b_1 - c + 1; b_1 - a + 1; t/x) \times F_1(1 + b_2 + b_3 - d, b_2, b_3; 1 + b_2 + b_3 - a'; (1-t)/y, (1-t)/z)\,dt, \qquad \ldots (3.8)$$

which is proportional to

$$x^{-b_1} y^{-b_2} z^{-b_3} \sum \frac{(b_1, m)\,(b_1 - c + 1, m)\,(1 + b_2 + b_3 - d, n + p)\,(b_2, n)\,(b_3, p)\,x^{-m} y^{-n} z^{-p}}{(2 - a - a' + b_1 + b_2 + b_3, m + n + p)\,m!\,n!\,p!} \quad \ldots (3.9)$$

$$= x^{-b_1} y^{-b_2} z^{-b_3} F_S(b_1 - c + 1, 1 + b_2 + b_3 - d, 1 + b_2 + b_3 - d, b_1, b_2, b_3;$$

$$2 - a - a' + b_1 + b_2 + b_3, 2 - a - a' + b_1 + b_2 + b_3, 2 - a - a' + b_1$$

$$+ b_2 + b_3;\ 1/x,\ 1/y,\ 1/z) \text{ (Saran}^4\text{)}. \quad \ldots (3.10)$$

This is the same as Φ_{7a} in the general list of Srivastava and Karlsson. Saran[5] gave the result

$$F_T(a_1, a_2, a_2, b_1, b_2, b_1; c, c, c; x, y, z)$$

$$= (1 - z)^{-a_2} F_S(a_1, a_2, a_2, b_1, b_2, c - b_1 - b_2;$$

$$c, c, c;\ x, (y - z)/(1 - z),\ z/(1 - z)), \quad \ldots (3.11)$$

which indicates that F_T is a special case of F_S.

We now replace the Appell function of (2.1) by further members of the list in Olsson[7] :

$$(y/(1 - t))^{-b_2} G_2(b_2, b_3, a' - b_2, 1 + b_2 - d;\ (t - 1)/y,\ z/(t - 1)), \quad \ldots (3.12)$$

where the function $G_2(a, b, c, d; x, y)$ is given by

$$\sum \frac{(a, m)\,(b, n)\,(c, n - m)\,(d, m - n)\,x^m y^n}{m!\,n!}. \quad \ldots (3.13)$$

We then have the further solution of the system F_G

$$y^{-b_2} \sum \frac{(b_1, m)(b_2, n)(b_3, p)(1 + b_2 - d, n - p)(a + a' - b_2 - 1, m + p - n)x^m(-y)^{-n}(-z)^p}{(c, m)m!\,n!\,p!}$$

$$= y^{-b_2} 4f(b_1, b_2, b_3, a + a' - b_2 - 1, 1 + b_2 - d; c; x, -z, -1/y), \quad \ldots (3.14)$$

which is the same as the function $D_3^{1,2}$ given by Exton[9].

Similarly, by replacing the ${}_2F_1$ function in (2.1) also, we obtain

$$x^{-b_1} y^{-b_2} \sum \frac{(b_1, m)(b_1 - c + 1, m)(b_2, n)(b_3, p)(1 + b_2 - d, n - p)(a + a' - 1 - b_1 - b_2, p - m - n)}{m!\,n!\,p!}$$

$$\times (-x)^{-m} (-y)^{-n} (-z)^p$$

$$= x^{-b_1} y^{-b_2} \Phi_{2d}(b_2, b_1, b_1 - c + 1, b_3, a + a' - 1 - b_1 - b_2,$$

$$1 + b_2 - d;\ -1/y, -1/x, z). \quad \ldots (3.15)$$

This last function is the same as G_D given first by Dhawan[11].

4. Further Second-order Solution Types of the System F_G

The solution types given in the previous section are not convergent near the origin. We now obtain similar results which converge in at least part of the neighbourhood of the origin. In the integral (2.1), replace the function F_1 in turn by

$$y^{b_3-d+1}\, z^{-b_3}\, (1-t)^{d-1}\, F_1\,(1+b_2+b_3-d,\, a'-d+1,\; b_3;\, b_3-d+2;$$

$$y/(1-t),\, y/z) \qquad \ldots (4.1)$$

and $$(z/(1-t))^{1-d}\, G_2\,(b_2,\, a'-d+1,\, b_3-d+1,\, d-1;\, -y/z,\, z/(t-1)). \quad \ldots (4.2)$$

In the first case, we obtain

$$y^{b_3-d+1}\, z^{-b_3} \times \sum \frac{(b_1, m)(1+b_2+b_3-d, n+p)\,(b_3, p)\,(a+a'-d, m+n)\, x^m\, y^n\, (y/z)^p}{(c, m)\,(b_3-d+2, n+p)\, m!\, n!\, p!}$$

$$= y^{b_3-d+1}\, z^{-b_3}\, F_M\,(b_1,\, 1+b_2+b_3-d,\, 1+b_2+b_3-d,\, a+a'-d,$$

$$b_3,\, a+a'-d;\, c,\, b_3-d+2,\, b_3-d+2;\, x,\, y/z,\, y). \quad \ldots (4.3)$$

This is the same as the function Φ_{11a} of Srivastava and Karlsson's list.

Similarly, (4.2) gives rise to

$$z^{1-d} \sum \frac{(b_1, m)(b_2, n)\,(b_3-d+1, p-m)(d-1, n-p)(a+a'-d, m+p)\, x^m (-y/z)^n\, (-z)^p}{(c, m)\, m!\, n!\, p!}$$

$$= z^{1-d}\, \Phi_{4h}\,(a+a'-d,\, b_1,\, b_2,\, b_3-d+1,\, d-1;\, c;\, -z,\, x,\, -y/z). \qquad \ldots (4.4)$$

No further solution types of the second order have been found which are associated with the partial differential system of F_G.

5. An Integral Representation for the Function F_T

Since the function F_T, or Φ_{13a} in Srivastava and Karlsson's list, is a special case of F_S, or Φ^*_{7a}, a separate investigation is of interest in view of the fact that F_S is associated with the system F_G. By comparison with the integral representation of a quadruple hypergeometric function, (see Exton[10]), the following integral is proposed :

$$F_T\,(c, c', c', b, b', b;\, d, d, d;\, x, y, z)$$

$$= \Gamma(1-b)\,\Gamma(1-b')\,\Gamma(b+b')\,(2\pi i)^{-2}$$

$$\times \int (-t)^{b-1}\,(t-1)^{b'-1}\, F_1\,(b+b',\, c,\, c';\, d;\, xt,\, y(1-t)+zt)\, dt, \quad \ldots (5.1)$$

in which the contour of integration is a Pochhemmer double loop slung around the points 0 and 1 along with the other contour integrals in this paper.

This expression may easily be established using term-by-term integration, since the series expansion of the integrand is uniformly convergent on the contour. The

integral (5.1) is rather different in character from those discussed by Saran[5]. As before, we replace the Appell function of the integrand by other appropriate solutions of the associated partial differential system as given by Olsson[7].

6. Second-order Solution Types of the System F_T

In the integral (5.1), replace the Appell function by

$$(xt)^{-c}\,[y(1-t)+zt]^{c-d+1}\,F_1\,(1+c+c'-d,\,c,\,b+b'-d+1;\,c-d+2;$$

$$(y(1-t)+zt)/xt,\;y(1-t)+zt), \qquad \ldots (6.1)$$

and consider the integral

$$x^{-c}\int(-t)^{b-1-c}\,(t-1)^{b'-1}\,[y(1-t)+zt]^{c-d+1}$$

$$\times\,F_1\,[1+c+c'-d,\,c,\,b+b'-d+1;\,c-d+2;$$

$$(y(1-t)+zt)/xt,\;y(1-t)+zt]. \qquad \ldots (6.2)$$

If the function F_1 is expanded as a double series, we have

$$x^{-c}\sum\frac{(1+c+c'-d,\,m+n)\,(c,\,m)\,(b+b'-d+1,\,n)\,(-x)^{-m}}{(c-d+2,\,m+n)\,m!\,n!}$$

$$\times\int(-t)^{b-1-c-m}\,(t-1)^{b'-1}\,[y(1-t)+zt]^{c-d+1+m+n}\,dt. \qquad \ldots (6.3)$$

The inner integrand may be expanded in two ways. The first yields the solution type

$$x^{-c}\,z^{c-d+1}$$

$$\times\sum\frac{(d-c-1,\,p-m-n)(1+b-d,\,n-p)(1+c+c'-d,\,m+n)(c,\,m)(b'\,p)(x/z)^m z^n (y/z)^p}{m!\,n!\,p!}$$

$$=x^{-c}\,z^{c-d+1}\,\Phi_{5f}\,(1+c+c'-d,\,c,\,b',\,1+b-d,\,d-c-1;\,x/z,\,z,\,y/z). \qquad \ldots (6.4)$$

The second expansion gives rise to a solution of the third order, which is not considered here.

Two more typical solutions follow by the use of

$$(xt)^{c'-d+1}\,[y(1-t)+zt]^{-c'}\,F_1\,(1+c+c'-d,\,1+b+b'-d,\,c';\,c'-d+2;$$

$$xt,\;xt/[y\,(1-t)+zt]) \qquad \ldots (6.5)$$

and

$$[y(1-t)+zt]^{1-d}\,G_2\,(c,\,1+b+b'-d,\,c'-d+1,\,d-1;$$

$$-xt/[y(1-t)+zt],\,y(t-1)-zt). \qquad \ldots (6.6)$$

These are, respectively,

$$x^{c'-d+1}z^{-c'}\,\Phi_{11c}(1+c+c'-d, c', b', 1+b-d:$$

$$c'-d+2; x, x/z, -y/z) \qquad \ldots (6.7)$$

and $$z^{1-d}\,\Phi_{4m}(c, b', d-1, c'-d+1,\ 1\ +\ b\ -\ d;\ -\ x/z,\ y/z,\ z). \qquad \ldots (6.8)$$

No further types of solutions of the second order have been deduced.

7. Conclusion

The second order solution types associated with F_G, or Φ_{18a} are Φ_{5b}, Φ_{5d}, F_S or Φ_{7a}, Φ_{4f}, Φ_{2d}, Φ_{4h} and F_M or Φ_{11a}. The function F_T is a special case of F_S, and is associated with Φ_{5f}, Φ_{11c} and Φ_{4m}. Furthermore, special cases of $D_{(3)}^{2,3}$ or Φ_{5d} and $D_{(3)}^{1,2}$ or Φ_{4f} occur in the study of the system of partial differential equations of $F_D^{(3)}$ or Φ_{19a}, which is, in turn, connected with $C_3^{(1)}$ or Φ_{5e}. Hence, the partial differential systems associated with each of the eight functions Φ_{18a}, Φ_{5b}, Φ_{5d}, Φ_{7d}, Φ_{4f}, Φ_{2d}, Φ_{4h} and Φ_{11a} are mutually equivalent. Similarly, so also are the systems of Φ_{13a}, Φ_{5f}, Φ_{11c} and Φ_{4m}, and the systems of Φ_{19a} and Φ_{5c}, respectively.

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References

1. H. M. Srivastava and P. W. Karlsson, Multiple Gaussian Hypergeometric Series, Ellis Horwood, Chichester, U.K., 1985, p. 74.
2. E. G. Kalnins, H. L. Manocha and Willard Miller (Jr.), *Stud. Appl. Math.* **62** (1980), 143-73.
3. G. Lauricella, *Rend. Circ. Mat. Palermo* **7** (1893), 111-58.
4. S. Saran, *Ganita* **5**, (1954), 77-91.
5. S. Saran, *Proc. Natn. Inst. Sci. India* **A21**, (1955), 83-90.
6. S. Saran, Corrigendum *Ganita* **7** (1956), 65.
7. P. O. M. Olsson, *J. Math. Phys.* **5** (1964), 420-30.
8. P. O. M. Olsson, *J. Math. Phys.* **18** (1977), 1285-94.
9. H. Exton, *Funkcial Ekvac.* **16** (1973), 189-94.
10. H. Exton, *Ann. Mat. Pura ed appl.* **161** (1992), 315-43.
11. G. K. Dhawan, *Proc. Natn. Acad. Sci. India* **A40** (1970), 43-48.