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INCLUSION THEOREMS FOR GENERALISED NÖRLUND SUMMABILITY METHODS

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This note concerns conditions for the generalised Nörlund summability method (N, p, q) to imply (N, u, q). Other results of interest are obtained as corollaries. In particular, the problem of the ineffectiveness of the $(\bar{N}, q)$ method for $q_n > 0$, is completely solved.

1. INTRODUCTION

Let (p_n), (q_n) and (u_n) be sequences of real or complex numbers, with $P_n = \sum_{v=0}^{n} p_v$, $Q_n = \sum_{v=0}^{n} q_v$, $r_n = \sum_{v=0}^{n} p_{n-v} q_v$, and $w_n = \sum_{v=0}^{n} u_{n-v} q_v$. The sequence (s_n) is generalised Nörlund summable to s, and we write $s_n \to s(N, p, q)$ if $t_n \to s$ as $n \to \infty$, where $t_n = \sum_{v=0}^{n} p_{n-v} q_v s_v / r_n$, and $r_n \neq 0$ for $n \geq 0$, $p_{-1} = 0$, etc. When $q_n = 1$ all n, the (N, p, q) method reduces to the Nörlund method (N, p), and when $p_n = 1$ all n, it reduces to the $(\bar{N}, q)$ method. Necessary and sufficient conditions for the (N, p, q) method to be regular, that is for $s_n \to s$ to imply $s_n \to s(N, p, q)$, are (i) $p_{n-v} q_v / r_n \to 0$ for each integer $v \geq 0$ as $n \to \infty$, and (ii) $\sum_{v=0}^{n} |p_{n-v} q_v| < H |r_n|$, where H is a positive number independent of n. In particular, necessary and sufficient conditions for (N, p) regularity, are $p_n = o(P_n)$ and $\sum_{v=0}^{n} |p_v| < H |P_n|$. Given Nörlund methods (N, p) and (N, q) generated by the sequences (p_n) and (q_n) respectively, we formally define $p(x) = \Sigma\, p_n x^n$, $q(x) = \Sigma\, q_n x^n$ and $k(x) = q(x)/p(x)$, so that $k(x)\, p(x) = q(x)$. It follows that $k(x)\, P(x) = Q(x)$, where $P(x) = \Sigma\, P_n x^n$ and $Q(x) = \Sigma\, Q_n x^n$ (see Hardy[4], § 4.3 for further details). We define the sequence of constants (c_n) formally by the identity $\Sigma\, c_n x^n \doteq (\Sigma\, p_n x^n)^{-1}$, $c_{-1} = 0$, and write $c_n^{(1)} = \sum_{v=0}^{n} c_v$. We write $(p_n) \in \mathcal{M}$ if $p_n > 0$,

$p_{n+1}/p_n \leq p_{n+2}/p_{n+1} \leq 1$ for all $n \geq 0$. The method A implies the method B if every sequence summable (A) is summable (B) to the same sum; that is, whenever $s_n \to s(A)$, then $s_n \to s(B)$. The methods are equivalent if each implies the other. A summability method is said to be ineffective if it is equivalent to convergence. We write $\Delta r_n = r_n - r_{n-1}$, and assume throughout that $q_n \neq 0$ all n. By K we denote a positive constant, not necessarily the same at each occurrence.

The purpose of this note is to study the relation between the (N, p, q) and (N, u, q) summability methods. As corollaries of the main results, we obtain conditions for the ineffectiveness of the (N, p, q) method and solve completely the problem of the ineffectiveness of the $(\bar{N}, q)$ method for $q_n > 0$.

2. Relations Between (N, p, q) and (N, u, q) Summability Methods

Theorem 1 — Let $(p_n) \in \mathcal{M}$, $q_n > 0$, $u_n > 0$ and (u_n) be non-decreasing Then (N, p, q) implies (N, u, q) if and only if

$$\sum_{\nu=0}^{n} u_{n-\nu} c_\nu = o(w_n). \qquad \ldots (2.1)$$

For the proof of this result, we require the following lemmas.

Lemma 1 (Das[3], Theorem 2) — Let p_0, u_0 and (q_n) be non-zero $(n \geq 0)$. Then (N, p, q) implies (N, u, q) if and only if $(N, \Delta r)$ implies $(N, \Delta w)$.

Remark : The first sentence of Lemma 1 is omitted in Das[3]. However, this cannot be omitted without making the result false. I am indebted to Professor Kuttner for this observation.

Lemma 2 (Borwein and Cass[1], Proposition 1) — For (N, p) to imply (N, q), it is necessary and sufficient that $\sum_{\nu=0}^{n} |k_{n-\nu} P_\nu| = O(|Q_n|)$ and $k_{n-\nu}/Q_n \to 0$ for each ν as $n \to \infty$.

Proof of Theorem : By Lemma 1, (N, p, q) implies (N, u, q) if and only if $(N, \Delta r)$ implies $(N, \Delta w)$. Writing $\sum_{\nu=0}^{n} l_{n-\nu} \Delta r_\nu = \Delta w_n$, so that $l_n = \sum_{\nu=0}^{n} \Delta u_{n-\nu} c_\nu^{(1)}$, since

$$\sum_{\nu=0}^{n} \left(\sum_{j=0}^{n-\nu} \Delta u_{n-\nu-j} c_j^{(1)} \sum_{m=0}^{\nu} \Delta p_{\nu-m} q_m \right) = \sum_{\nu=0}^{n} \Delta u_{n-\nu} q_\nu = \Delta w_n,$$

it follows from Lemma 2 that $(N, \Delta r)$ implies $(N, \Delta w)$ if and only if $\sum_{\nu=0}^{n} |l_{n-\nu} r_\nu| = O(|w_n|)$ and $l_{n-\nu} = o(w_n)$ for each ν as $n \to \infty$. Now

$$\sum_{\nu=0}^{n} |l_{n-\nu} r_\nu| = \sum_{\nu=0}^{n} \left| \sum_{j=0}^{n-\nu} \Delta u_{n-\nu-j} c_j^{(1)} \right| |r_\nu|$$

$$= \sum_{\nu=0}^{n} \sum_{j=0}^{n-\nu} \Delta u_{n-\nu-j}\, c_j^{(1)} r_\nu = \sum_{\nu=0}^{n} \sum_{j=0}^{n-\nu} u_{n-\nu-j}\, c_j r_\nu = \sum_{\nu=0}^{n} u_{n-\nu} q_\nu = w_n, \text{ since } c_j^{(1)} \geq 0$$

because $(p_n) \in \mathcal{M}$ (see Hardy[4], Theorem 22), and $\Delta u_n \geq 0$. The second condition requires that

$$\sum_{j=0}^{n-\nu} \Delta u_{n-\nu-j}\, c_j^{(1)} = o(w_n) \qquad \text{... (2.2)}$$

for each ν as $n \to \infty$. Since (w_n) is positive and non-decreasing, (2.2) if true for $\nu = 0$, is true for all $\nu > 0$, and hence may be written as $\sum_{\nu=0}^{n} \Delta u_{n-\nu} c_\nu^{(1)} = o(w_n)$, which is the same as (2.1).

Corollary 1 — Let $(p_n) \in \mathcal{M}$, $q_n > 0$, $u_n > 0$ and (u_n) be non-decreasing. Then (N, p, q) implies (N, u, q) if and only if (i) at least one of the sequences (P_n) or (Q_n) is unbounded, and (ii) $(N, \Delta u, Q)$ is regular.

PROOF : From Theorem 1, we have (2.1), which can be written

$$\sum_{\nu=0}^{n} \Delta u_{n-\nu} Q_\nu \,(c_\nu^{(1)}/Q_\nu) = o\left(\sum_{\nu=0}^{n} \Delta u_{n-\nu} Q_\nu \right). \qquad \text{... (2.3)}$$

Now $(c_n^{(1)}/Q_n)$ is non-increasing and positive; so by (ii), (2.3) holds if and only if $c_n^{(1)} = o(Q_n)$. Then either $c_n^{(1)} = o(1)$ (in which case $P_n \to \infty$) or $Q_n \to \infty$ (or both). If (Q_n) and (P_n) are both bounded, so that $(c_n^{(1)})$ has a non-zero limit, then (2.3) cannot be satisfied. Also, from (2.3), as all terms are positive, then $\Delta u_{n-\nu} = o(w_n)$ for each ν as $n \to \infty$. Thus $(N, \Delta u, Q)$ is regular.

Remark : The condition (i) of Corollary 1 may be replaced by $R_n \to \infty$. This follows from $P_0 Q_n \leq \sum_{\nu=0}^{n} P_{n-\nu} q_\nu = R_n \leq P_n Q_n$ and $Q_0 P_n \leq \sum_{\nu=0}^{n} p_{n-\nu} Q_\nu = R_n$.

Corollary 2 — Let $(p_n) \in \mathcal{M}$ and $q_n > 0$. Then (N, p, q) implies $(\overline{N}, q)$ if and only if at least one of the sequences (P_n) or (Q_n) is unbounded.

Remark : The sufficiency part of this corollary is Theorem 4 of Das[2].

Corollary 3 — Let (p_n) and $(p_n') \in \mathcal{M}$, $q_n > 0$ and $(R_n), (R_n')$ be unbounded. Then the methods (N, p, q) and (N, p', q) are consistent.

Here, $R_n' = \sum_{\nu=0}^{n} p_{n-\nu}' Q_\nu$. Recall that two methods are consistent if they cannot sum the same sequence to different limits. The result follows from Corollary 2 by noting that if the methods did sum the same sequence to different limits, then so would the $(\overline{N}, q)$ method.

Corollary 4 — If $q_n > 0$, then (N, q) implies (N, Q) if and only if $Q_n^{(1)} \to \infty$.

PROOF : Put $u_n = P_n$ in Theorem 1 and apply Lemma 1. Thus with $(p_n) \in \mathcal{M}$, $q_n > 0$, then $(N, \Delta r)$ implies (N, r) if and only if $R_n \to \infty$. Now put $p_n = 1$ all n.

Remark : For an extension of the sufficiency half of this result to real sequences (q_n), see Corollary 3 of Thorpe[6].

Corollary 5 — Let $(p_n) \in \mathcal{M}$, and (u_n) be positive and non-decreasing. Then (N, p) implies (N, u) if and only if $\Delta u_n = o(U_n)$.

PROOF : Put $q_n = 1$ all n, in Corollary 1.

Remark : We may replace $\Delta u_n = o(U_n)$ here by (N, u) regularity, since with the conditions of Corollary 5, $\Delta u_n = o(U_n)$ is equivalent to $u_n = o(U_n)$.

Corollary 6 — Let (p_n) be positive and non-decreasing. Then (C, k) implies (N, p) if and only if (N, p) is regular, $0 \leq k \leq 1$.

Remarks : The case $k = 0$ of this result requires separate treatment. A special case of Corollary 6 is Hardy's[4] Theorem 20. For an extension of the latter to complex (p_n), see Theorem 5 of Thorpe[6].

Corollary 7 — If $(p_n) \in \mathcal{M}$, then (N, p) implies (C, k), $k \geq 1$.

Corollary 8 — Let (p_n) and (q_n) be positive sequences with (p_n) non-decreasing. Then $(\bar{N}, q)$ implies (N, p, q) if and only if $\Delta p_n = o(r_n)$.

Theorem 2 — Let (p_n) and (q_n) be positive sequences with (p_n) non-increasing. (i) $(\bar{N}, q)$ implies (N, p, q) if and only if $Q_n = O(r_n)$. (ii) If $q_n = o(Q_n)$, then $(\bar{N}, q)$ implies (N, p, q) if and only if (p_n) converges to a non-zero limit.

PROOF : (i) By Lemma 1, $(\bar{N}, q)$ implies (N, p, q) if and only if (N, q) implies $(N, \Delta r)$. Writing $\sum_{\nu=0}^{n} \Delta p_{n-\nu} q_\nu = \Delta r_n$, it follows by Lemma 2 that (N, q) implies $(N, \Delta r)$ if and only if $\sum_{\nu=0}^{n} |\Delta p_{n-\nu}| Q_\nu = O(|r_n|)$ and $\Delta p_{n-\nu} = o(r_n)$ for each ν as $n \to \infty$. Since (p_n) is non-increasing, $\sum_{\nu=0}^{n} |\Delta p_{n-\nu}| Q_\nu = \sum_{\nu=0}^{n} |p_{n-\nu-1} - p_{n-\nu}| Q_\nu = 2p_0 Q_n - r_n = O(|r_n|)$, if and only if $Q_n = O(r_n)$. Also, (p_n) converges; so $\Delta p_{n-\nu} \to 0$ as $n \to \infty$, for each ν. As $Q_n = O(r_n)$ is evidently meaningful, then we may discard the condition $\Delta p_{n-\nu} = o(r_n)$. (ii) From (i), a necessary and sufficient condition is $Q_n = O(r_n)$. Now $r_n/Q_n = \sum_{\nu=0}^{n} q_{n-\nu} p_\nu / Q_n$ has a non-zero limit if and only if (p_n) does, as (N, q) is regular. Since (p_n) converges, because it is non-increasing, the result follows.

Remark : Corollary 8 and Theorem 2 may be compared with Theorem 3 of Das[3] where (p_n) is monotone in either sense, but the conditions are only proved to be sufficient for $(\bar{N}, q)$ to imply (N, p, q).

Corollary 9 — If (p_n) is positive and non-increasing, then $(C, 1)$ implies (N, p) if and only if (p_n) converges to a non-zero limit.

3. Equivalence and Ineffectiveness of the $(\bar{N}, q)$ and (N, p, q) Methods

The results of the last section lead to a number of additional corollaries of independent interest, and we collect these here. In particular, we can solve completely the problem of the ineffectiveness of the $(\bar{N}, q)$ method for $q_n > 0$. We also obtain some conditions for the ineffectiveness of the (N, p, q) method and for the equivalence of the two methods.

Corollary 10 — If $q_n > 0$, then $s_n \to s(\bar{N}, q)$ implies $s_n \to s$ if and only if $Q_n = O(q_n)$.

PROOF : Put $p_0 = 1$, $p_n = 0$ $(n > 0)$, in Theorem 2(i).

Remark : It follows from Corollary 10, that $Q_n = O(q_n)$ is a necessary condition for the $(\bar{N}, q)$ method to be ineffective. Also, when $q_n > 0$, it is known that the $(\bar{N}, q)$ method is regular if and only if $Q_n \to \infty$. Now it is easy to see that if $Q_n = O(q_n)$, then $Q_n \to \infty$. Thus if $Q_n = O(q_n)$, then $(\bar{N}, q)$ is regular. Combining this with the sufficiency half of Corollary 10 (compare Hardy[4], Theorem 15), we have that if $Q_n = O(q_n)$, then $(\bar{N}, q)$ is equivalent to convergence. Collecting these results together, gives :

Corollary 11 — If $q_n > 0$, then the $(\bar{N}, q)$ method is ineffective if and only if $Q_n = O(q_n)$.

The next three results give conditions for the equivalence of the $(\bar{N}, q)$ and (N, p, q) methods.

Corollary 12 — Let $(p_n) \in \mathcal{M}$, and $q_n > 0$. Then (N, p, q) is equivalent to $(\bar{N}, q)$ if and only if (i) $Q_n = O(r_n)$ and (ii) at least one of (P_n) or (Q_n) is unbounded.

PROOF : From Corollary 2 and Theorem 2(i).

Corollary 13 — Let $(p_n) \in \mathcal{M}$, $q_n > 0$ and $q_n = o(Q_n)$. Then (N, p, q) is equivalent to $(\bar{N}, q)$ if and only if (p_n) converges to a non-zero limit.

PROOF : The necessity of the condition follows from Theorem 2(ii). For sufficiency, $P_n \to \infty$ and $r_n \geq p_n Q_n$, so $Q_n = O(r_n)$. Now use Corollary 12.

Corollary 14 — Let $(p_n) \in \mathcal{M}$, (P_n) converge and (q_n) be positive and non-decreasing. Then (N, p, q) is equivalent to $(\bar{N}, q)$ if and only if $Q_n = O(q_n)$.

PROOF : Plainly, $Q_n = O(q_n)$ implies $Q_n = O(r_n)$, and since $r_n \leq q_n P_n \leq K q_n$, then $Q_n = O(r_n)$ implies $Q_n = O(q_n)$. Now use Corollary 12.

Concerning (N, p, q) ineffectiveness, Das[3] says that it can be deduced from his Theorem 2(d) combined with Theorem 15 of Hardy[4]. This seems to be impossible. For the former result requires (N, q) to be regular, a necessary condition for which is $q_n = o(Q_n)$, while the latter requires $Q_n = O(q_n)$, as indicated above. As these conditions are mutually exclusive, no result can be obtained. However, as shown in Corollary 9 of Nurcombe[5], the assumption of (N, q) regularity in Theorem 2(d) of Das[3] is unnecessary, and a result on (N, p, q) ineffectiveness can be obtained as follows :

Corollary 15 — Let $(N, \Delta r)$ be regular, $\Sigma | c_n^{(1)} |$ and $\Sigma | \Delta p_n |$ converge, $q_n > 0$ and $Q_n = O(q_n)$. Then (N, p, q) is ineffective.

Theorem 3 — Let $(p_n) \in \mathcal{M}$, and $q_n > 0$. Then (N, p, q) is ineffective if and only if $p_n = o(q_n)$ and $r_n = O(q_n)$.

PROOF : A necessary and sufficient condition for (N, p, q) regularity is $p_{n-\nu} = o(r_n)$ for each ν as $n \to \infty$ (see §1). If $p_n = o(q_n)$, then $p_n = o(r_n)$, and $p_{n-\nu} = p_{n-\nu}\, p_n / p_n \leq K p_n = o(r_n)$, since $p_{n-\nu}/p_n \leq K$ for each ν, as $(p_n) \in \mathcal{M}$. Necessary and sufficient conditions for (N, p, q) to imply convergence are, by analogy with those for (N, p, q) regularity, $\sum_{\nu=0}^{n} | c_{n-\nu} r_\nu | = O(| q_n |)$ and $c_{n-\nu} = o(q_n)$ for each ν as $n \to \infty$. Since $(p_n) \in \mathcal{M}$, then $\sum_{\nu=0}^{n} | c_{n-\nu} r_\nu | = c_0 r_n - \sum_{\nu=0}^{n-1} c_{n-\nu} r_\nu = 2c_0 r_n - q_n = O(q_n)$ if and only if $r_n = O(q_n)$. Since (N, p, q) is regular, then $c_{n-\nu} = o(r_n)$, [see Nurcombe[5], Corollary 2(a)], and as $r_n = O(q_n)$, then $c_{n-\nu} = o(q_n)$. Thus the conditions are sufficient for (N, p, q) ineffectiveness. Conversely, if (N, p, q) is ineffective, then $r_n = O(q_n)$ and $p_{n-\nu} = o(r_n)$, as shown. When $\nu = 0$, we have $p_n = o(r_n)$ implying $p_n = o(q_n)$, as required.

Corollary 16 — Let $(p_n) \in \mathcal{M}$, $q_n > 0$ and (p_n) converge to a non-zero limit. Then (N, p, q) is ineffective if and only if $Q_n = O(q_n)$.

PROOF : In view of the inequalities $p_n Q_n \leq r_n \leq p_0 Q_n$, and (p_n) converging to a positive limit, it follows that $r_n = O(q_n)$ and $Q_n = O(q_n)$ are equivalent. Since the latter implies that $q_n \to \infty$, then $p_n = o(q_n)$, and the result now follows from Theorem 3.

Corollary 17 — Let $(p_n) \in \mathcal{M}$, $q_n > 0$ and $P_n \to \infty$. Then (N, p, q) is ineffective if and only if $r_n = O(q_n)$ and $Q_n \to \infty$.

PROOF : The sufficiency of the conditions follows from Theorem 5 of Nurcombe[5]. Conversely, if (N, p, q) is ineffective, then $p_n = o(q_n)$ by Theorem 3. Now $q_n/p_n \to \infty$, so that $Q_n \to \infty$.

Corollary 18 — Let $(p_n) \in \mathcal{M}$, (P_n) converge and $nq_n \geq K > 0$. Then (N, p, q) is ineffective if and only if $r_n = O(q_n)$.

PROOF : Since $nq_n \geq K > 0$, then $p_n/q_n \leq np_n/K = o(1)$, as $np_n = o(1)$ because Σp_v is a convergent series of positive, non-increasing terms. The conclusion now follows from Theorem 3.

Remark : Examples are readily constructed to show that when (P_n) converges, $Q_n \to \infty$ is not a necessary condition for (N, p, q) ineffectiveness.

Other results on (N, p, q) ineffectiveness can be found in Nurcombe[5].

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