Indian J. pure appl. Math., 25(10): 1089-1097, October 1994

EFFECT OF NON-HOMOGÉNEITY ON ELASTIC-PLASTIC TRANSITION IN A THIN ROTATING DISC

S. K. GUPTA AND R. K. SHUKLA

Department of Mathematics, Himachal Pradèsh University, Shimla 171 005

(*Received* 14 *June* 1993; *accepted* 24 *May* 1994)

Effect of non-homogeneity on elastic-plastic transition in a thin rotating annular disc has been investigated by using Seth's transition theory. The non-homogeneity is assumed due to the variation of modulus of rigidity. Expressions for stresses and angular velocity required for initial yielding and fully plastic state have been obtained. As a numerical example, it has been seen that the presence of non-homogeneity ($k > 0$) in thin rotating discs require higher angular velocity for initial yielding as compared to homogeneous disc but less percent increase in angular velocity to become fully plastic against initial yielding and this percentage goes on decreasing with the increase in non-homogeneity. Reverse is the case for $k < 0$.

NOMENCLATURE

u, v, w = displacement components

r, θ, z = cylindrical polar co-ordinates

$e_{rr}, e_{\theta\theta}, e_{zz}, \ldots$ = strain components

a, b = inner and outer radii

$\tau_{rr}, \tau_{\theta\theta}, \tau_{zz}, \ldots$ = stress components

$\mu(r)$ = variable modulus of rigidity

μ_0 = constant modulus of rigidity

ν = Poisson's ratio

n = measure

ρ = density

ω = angular velocity

k = a constant

λ = Lame's constant

β = function of r and $r\beta' = \beta p$

c = function of r and $c = 2\mu/(\lambda + 2\mu)$

c_0	= a constant and $c_0 = 2\mu_0/(\lambda + 2\mu_0)$
$f_1(r), I_1(r)$	= functions of r
R	= transition function
A_1, A_2	= constants of integration
Y	= yield stress
W_1	= speed factor for initial yielding
W_2	= speed factor for fully plastic state
N	= non-homogeneity ratio (bore to rim)
P_1	= percent increase in angular velocity required for initial yielding by a non-homogeneous disc against homogeneous disc.
P_2	= percent increase in angular velocity required by discs to become fully plastic against initial yielding.

1. INTRODUCTION

Helicopter rotor blades are typically built-up, composite structures and made of materials that may be anisotropic and non-homogeneous. For wide class of materials such as hot rolled copper, aluminium and magnesium alloys some degree of non-homogeneity is present. Olszak and Urbanowski[10] solved the problem of a thick walled cylinder, non-homogeneous both elastically and plastically subjected to internal and external pressures and showed that plastic flow may start from either surface depending on the character and intensity of the non-homogeneity. Ghosh[2] worked on the problem involving the study of elastico-plastic stresses in a spherical pressure vessel of non-homogeneous material and Mukhopadhyay[9] studied the effect of non-homogeneity on yield stress in a thick walled cylindrical tube under pressure. Recently, Erguven[1] has derived the axisymmetric fundamental solution for non-homogeneous transversely isotropic media by using the Hankel integral transform.

Turbine discs in modern aero-engines rotate at high speed and, as a consequence, are subjected to high stress. In this paper, the problem of a thin rotating disc of non-homogeneous material has been solved by utilizing Seth's transition theory[11, 13]. The theory does not require any ad hoc assumptions like an yield criterion, incompressibility condition etc. and thus poses and solves a more general problem from which cases pertaining to these assumptions can be worked out. It utilizes the concept of generalized strain measure and asymptotic solution at turning points or transition points of the governing differential equations defining the deformed field and has successfully been applied to a large number of problems in plasticity and creep[3-8].

The rigidity modulus of a thin rotating disc is assumed to vary radially i.e.

$$\mu = \mu_0 \bar{r}^k \quad \text{... (1)}$$

where μ_0 and k are real constants.

The elastic-plastic transitional stresses and the angular velocity needed for initial yielding have been found out. Results obtained have been compared with that of homogeneous disc and are discussed numerically.

2. Governing Equations

Consider an annular disc of internal radius 'a' and external radius 'b' rotating with an angular velocity ω about an axis perpendicular to its plane and passing through the centre. The disc is made of the material having constant density ρ but variable modulus of rigidity $\mu = \mu(r)$. The thickness of the disc is assumed sufficiently small so that it is effectively in a state of plane stress, that is, the axial stress τ_{zz} is zero. The components of displacement in cylindrical polar co-ordinates are given by[12]

$$u = r(1-\beta)\,,\ v = 0,\ w = dz \qquad \dots (2)$$

where β is a function of $r = \sqrt{x^2+y^2}$ only and d is a constant.

The generalized components of strain are obtained as[13]

$$\left.\begin{aligned} e_{rr} &= \frac{1}{n}\,[1-(r\beta'+\beta)^n],\\ e_{\theta\theta} &= \frac{1}{n}\,[1-\beta^n],\\ e_{zz} &= \frac{1}{n}\,[1-(1-d)^n],\\ e_{r\theta} &= e_{\theta z} = e_{zr} = 0 \end{aligned}\right] \qquad \dots (3)$$

where n is the measure and $\beta' = d\beta/dr$.

The stress = strain relations are[15]

$$\left.\begin{aligned} \tau_{rr} &= \frac{2\mu\lambda}{\lambda+2\mu}\,[e_{rr}+e_{\theta\theta}] + 2\mu\, e_{rr},\\ \tau_{\theta\theta} &= \frac{2\mu\lambda}{\lambda+2\mu}\,[e_{rr}+e_{\theta\theta}] + 2\mu\, e_{\theta\theta},\\ \tau_{zz} &= \tau_{zr} = \tau_{r\theta} = \tau_{\theta z} = 0 \end{aligned}\right] \qquad \dots (4)$$

where λ is a constant and $\mu = \mu(r)$.

Substituting from eqns. (3) in eqns. (4), the non-zero stress components are

$$\left.\begin{aligned} \tau_{rr} &= \frac{2\mu}{n}\,[3-2c-\beta^n\,\{1-c+(2-c)\,(P+1)^n\}],\\ \tau_{\theta\theta} &= \frac{2\mu}{n}\,[3-2c-\beta^n\,\{2-c+(1-c)\,(P+1)^n\}] \end{aligned}\right] \qquad \dots (5)$$

where

$$c = 2\mu/(\lambda+2\mu) \text{ and } r\beta' = \beta P.$$

All equations of equilibrium are satisfied except,

$$\frac{d(r\tau_{rr})}{dr} - \tau_{\theta\theta} + \rho\,\omega^2 r^2 = 0. \qquad \text{... (6)}$$

Substituting eqns. (5) in eqn. (6), we get a non-linear differential equation in β as

$$\begin{aligned} n(2-c)\,\mu\beta^{n+1} P(P+1)^{n-1}\, dP/d\beta \\ = r\mu'\,[3-2c-\beta^n\ \{1-c+(2-c)\,(P+1)^n\}]\ \frac{n\rho\omega^2 r^2}{2} \\ -\ \mu[2rc' - c'r\,\beta^n\,\{1+(P+1)^n\}\ + n\beta^n P\ \{1-c+(2-c)\,(P+1)^n\}] \\ +\ \mu\beta^n\,[1-(P+1)^n]. \end{aligned} \qquad \text{... (7)}$$

The transition points of β in eqn. (7) are $P = -1$ and $P = \pm\infty$. The boundary conditions are

$$\tau_{rr} = 0 \quad \text{at } r = a \text{ and } r = b. \qquad \text{... (8)}$$

3. Solution

It has been shown[5,6,8,14] that the asymptotic solution through the principal stress gives the plastic stresses at the transition point $P = \pm\infty$.

We take the transition function R as

$$R = \tau_{\theta\theta} \equiv 2\mu[3-2c-\beta^n\ \{2-c+(1-c)\,(P+1)^n\}]/n. \qquad \text{... (9)}$$

Taking the logarithmic differentiation of eqn. (9) with respect to r, we get

$$\frac{d\,(\log R)}{dr} = -\frac{\begin{array}{c} 2c'r - c'r\beta^n\{1+(P+1)^n\} + n\beta^n P\,\{2-c+(1-c)(P+1)^n\} \\ + n(1-c)\beta^{n+1}P(P+1)^{n-1}\,dP/d\beta \end{array}}{r[3-2c-\beta^n\{2-c+(1-c)(P+1)^n\}]} + \frac{\mu'}{\mu}. \qquad \text{... (10)}$$

Substituting the value of $dP/d\beta$ from eqn. (7) in eqn. (10) and taking the asymptotic value as $P \to \pm\infty$, we get after integration

$$R = A_1\left(\frac{1-c}{2-c}\right)\exp f_1(r) \qquad \text{... (11)}$$

where A_1 is a constant of intergration and

$$f_1(r) = -\int [1/r(2-c)]\, dr.$$

From eqns. (9) and (11), we have

$$\tau_{\theta\theta} = A_1\left(\frac{1-c}{2-c}\right)\exp f_1(r). \qquad \text{... (12)}$$

Substituting eqn. (12) in eqn. (6) and integrating, we get

$$r\,\tau_{rr} = A_2 + A_1 \int I_1(r)\,dr - \rho\,\omega^2 r^3/3 \qquad \dots (13)$$

where A_2 is a constant of integration and

$$I_1(r) = \left(\frac{1-c}{2-c}\right) \exp f_1(r).$$

Using boundary conditions (8) in eqn. (13), we obtain

$$A_1 = \frac{\rho\omega^2}{3}\left[\frac{b^3 - a^3}{\int_a^b I_1(r)\,dr}\right],\quad A_2 = \frac{\rho\omega^2}{3}\left[a^3 - \left(\frac{b^3 - a^3}{\int_a^b I_1(r)\,dr}\right)\int^a I_1(r)\,dr\right]. \qquad \dots (14)$$

Substituting the values of A_1 and A_2 in eqn. (13), we get

$$\tau_{rr} = \frac{\rho\omega^2}{3r}\left[\left(\frac{b^3 - a^3}{\int_a^b I_1(r)\,dr}\right)\int_a^r I_1(r)\,dr - r^3 + a^3\right] \qquad \dots (15)$$

and eqn. (12) becomes

$$\tau_{\theta\theta} = \frac{\rho\omega^2}{3}\left[\frac{b^3 - a^3}{\int_a^b I_1(r)\,dr}\right] I_1(r). \qquad \dots (16)$$

Equations (15) and (16) give elastic-plastic transitional stresses in a thin rotating disc made of non-homogeneous material.

Substituting eqn. (1) in eqns. (15) and (16), we get

$$\tau_{rr} = \frac{\rho\omega^2}{3r}\left[\frac{(b^3 - a^3)}{\left\{(r^k + \mu_0/\lambda)^{1/2k}\right\}_a^b}\left\{(r^k + \mu_0/\lambda)^{1/2k}\right\}_a^r - r^3 + a^3\right],$$

$$\tau_{\theta\theta} = \frac{\rho\omega^2 (b^3 - a^3)}{6r^{1-k}\left\{(r^k + \mu_0/\lambda]^{1/2k}\right\}_a^b}(r^k + \mu_0/\lambda)^{(1/2k)-1}. \qquad \dots (17)$$

It has been seen from eqns. (17) that $|\tau_{\theta\theta}|$ has the greatest value at $r = a$. Therefore, yielding in the disc will start at the inner radius and in this case we have

$$|\tau_{\theta\theta}|_{r=a} = \left| \frac{\rho\omega^2 (b^3 - a^3)(a^k + \mu_0/\lambda)^{(1/2k)-1}}{6a^{1-k}\left\{(r^k + \mu_0/\lambda)^{1/2k}\right\}_a^b} \right| \equiv Y \text{ (say)}. \quad \ldots (18)$$

The angular velocity ω_i necessary for initial yielding is given by

$$\frac{\rho\omega_i^2 b^2}{Y} = \frac{6a^{1-k}\left\{(r^k + \mu_0/\lambda)^{1/2k}\right\}_a^b}{\left\{1 - (a/b)^3\right\} b(a^k + \mu_0/\lambda)^{(1/2k)-1}} \equiv W_1 \quad \ldots (19)$$

or

$$\omega_i = \frac{1}{b}(W_1 Y/\rho)^{1/2}. \quad \ldots (20)$$

The angular velocity $\omega_f (> \omega_i)$ at which the disc becomes fully plastic, $(\lambda \to \infty)$ is obtained from eqn. (17) as,

$$\frac{\rho\omega_f^2 b^2}{Y} = 6\left[\frac{1 - \sqrt{a/b}}{1 - (a/b)^3}\right] \equiv W_2 \quad \ldots (21)$$

or

$$\omega_f = \frac{1}{b}(W_2 Y/\rho)^{1/2}. \quad \ldots (22)$$

The stresses (17) for fully plastic state $(\lambda \to \infty)$ become

$$\tau_{rr} = \frac{\rho\omega_f^2}{3r}\left[\left(\frac{b^3 - a^3}{\sqrt{b} - \sqrt{a}}\right)(\sqrt{r} - \sqrt{a}) - r^3 + a^3\right],$$

$$\tau_{\theta\theta} = \frac{\rho\omega_f^2}{6\sqrt{r}}\left(\frac{b^3 - a^3}{\sqrt{b} - \sqrt{a}}\right). \quad \ldots (23)$$

4. HOMOGENEOUS CASE

For homogeneous material (i.e. $k = 0$) we have from eqn. (1), $\mu = \mu_0$ (a constant). Consequently, eqn. (11) becomes

$$R = A_1 r^{\nu - 1} \quad \ldots (24)$$

where $\nu = (1 - c_0)/(2 - c_0)$ is the Poisson's ratio and $c_0 = 2\mu_0/(\lambda + 2\mu_0)$. The elastic-plastic transitional stresses (15) and (16) become

$$\tau_{rr} = \frac{\rho\omega^2}{3r}\left[\left(\frac{b^3 - a^3}{b^\nu - a^\nu}\right)(r^\nu - a^\nu) - r^3 + a^3\right],$$

$$\tau_{\theta\theta} = \frac{\rho\omega^2 \nu}{3r^{1-\nu}}\left[\frac{b^3 - a^3}{b^\nu - a^\nu}\right] \quad \ldots (25)$$

yielding in the disc starts at the inner radius and

$$|\tau_{\theta\theta}|_{r=a} = \left| \frac{\rho\omega^2 \nu}{3a^{1-\nu}} \left(\frac{b^3 - a^3}{b^\nu - a^\nu} \right) \right| = Y. \quad \ldots (26)$$

The angular velocity necessary for initial yielding is

$$\frac{\rho\omega_i^2 b^2}{Y} = \frac{3(a/b)^{1-\nu}}{\nu} \left[\frac{1-(a/b)^\nu}{1-(a/b)^3} \right]. \quad \ldots (27)$$

For fully plastic state i.e. $\nu = 1/2$, the stresses (25) become

$$\tau_{rr} = \frac{\rho\omega_f^2}{3r} \left[\left(\frac{b^3 - a^3}{\sqrt{b} - \sqrt{a}} \right) (\sqrt{r} - \sqrt{a}) - r^3 + a^3 \right],$$

$$\tau_{\theta\theta} = \frac{\rho\omega_f^2}{6\sqrt{r}} \left[\frac{b^3 - a^3}{\sqrt{b} - \sqrt{a}} \right] \quad \ldots (28)$$

where

$$\rho\omega_f^2 b^2/Y = 6 \left[\frac{1-\sqrt{a/b}}{1-(a/b)^3} \right].$$

It can be seen that eqns. (23) and (28) for non-homogeneous as well as for homogeneous material are identical for fully plastic state. For homogeneous material, expressions (28) are the same as obtained by Gupta and Shukla[8].

For numerical computations, we have taken the value of the constant c_0 to be 0.50746 so that the Poisson's ratio $\nu = (1-c_0)/(2-c_0)$ for homogeneous material comes out to be 0.33. The materials under this range of Poisson's ratio are hot-rolled copper, cold drawn brass etc. We have considered the following two types of discs as shown in Table I :

Case I — Discs having less non-homogeneity at the bore than at the rim, i.e. $k < 0$.

Case II — Discs having more non-homogeneity at the bore than at the rim, i.e. $k > 0$.

Angular velocity necessary for initial yielding and fully plastic state has been calculated from eqns. (19), (21) and (27) and the results obtained are listed in Tables II, III and IV.

From Table II, it can be observed that case II disc ($k > 0$) requires higher angular velocity to yield and case I ($k < 0$) requires less angular velocity to yield as compared to disc made of homogeneous material and this speed goes on increasing for case II with the increase in values of k whereas it decreases for case I with higher negative values of k.

Table III gives the percent increase (or decrease) in angular velocity required for initial yielding of a non-homogeneous disc in case II (or case I) against homogeneous disc.

Table IV shows the percent increase in angular velocity required by a disc to become fully plastic against the initial yielding. It can be seen that a case II disc requires less percent increase in angular velocity to become fully plastic from its initial yielding as compared to the homogeneous disc and reverse is the case for case I, that is, higher percent increase in angular velocity is required to become fully plastic. This percentage becomes less and less for case II as k increases whereas it increases for case I as k takes higher negative values. However, a disc having higher radii ratio, less percentage of angular velocity is required to become fully plastic from the initial yielding.

TABLE I

Variation of $N = \mu_a/\mu_b$ for different discs, $\mu = \mu_0 \bar{r}^k$

$(a/b)\backslash k$	− 1.0	− 1/2	0	1/6	1/4	1/2	1.0	2.0
0.10	0.10	0.31623	1.0	1.46780	1.77828	3.16228	10.0	100.0
0.26	0.26	0.50990	1.0	1.25171	1.40041	1.96116	3.84615	14.79290
0.50	0.50	0.70771	1.0	1.12246	1.18921	1.41421	2.0	4.0

TABLE II

Angular velocity required for initial yielding and fully plastic state

	Initial yielding state $W_1 = \frac{\rho\omega_i^2 b^2}{Y}$								Fully plastic state $W_2 = \frac{\rho\omega_f^2 b^2}{Y}$
$(a/b)\backslash k$	− 1.0	− 1/2	0	1/6	1/4	1/2	1.0	2.0	
0.10	0.51817	0.74844	1.03555	1.12990	1.17497	1.29867	1.48729	1.68708	4.10674
0.26	0.95777	1.15032	1.34675	1.40932	1.43968	1.52624	1.67597	1.88349	2.99320
0.50	1.15166	1.24485	1.33513	1.36416	1.37843	1.42016	1.49823	1.63052	2.00841

TABLE III

Percent increase in angular velocity required for initial yielding by a non-homogeneous disc against homogeneous disc

$$P_1 = [(W_1^k/W_1^0)^{1/2} - 1] \times 100$$

$(a/b)\backslash k$	− 1.0	− 1/2	1/6	1/4	1/2	1.0	2.0
0.10	− 29.26255	− 14.98564	4.45651	6.51959	11.98602	19.84327	27.63894
0.26	− 15.66893	− 7.58002	2.29681	3.39251	6.45541	11.55532	18.26009
0.50	− 7.12483	− 3.44043	1.08116	1.60841	3.13505	5.93200	10.50973

TABLE IV

Percent increase in angular velocity required by discs to become fully plastic against initial yielding

$P_2 = [(W_2/W_1)^{1/2} - 1] \times 100$

$(a/b) \backslash k$	− 1.0	− 1/2	0	1/6	1/4	1/2	1.0	2.0
0.10	181.52334	134.24567	99.14243	90.64626	86.95381	77.82793	66.16904	56.02012
0.26	76.78150	61.30893	49.08173	45.73448	44.19007	40.04147	33.63928	26.06258
0.50	32.05783	27.01893	22.64892	21.33708	20.70745	18.92071	15.78080	10.98473

CONCLUSION

The presence of non-homogeneity of case II material in thin rotating annular discs therefore requires greater angular velocity for initial yielding and less percent increase in angular velocity to become fully plastic from the initial yielding as compared to disc made of homogeneous material and reverse is the case for case I discs. The use of non-homogeneous material may therefore be beneficial for the manufacture of disc components as they may provide a longer service life than homogeneous disc components under identical conditions.

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