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EXISTENCE OF PERIODIC ORBITS OF FIRST KIND IN THE CIRCULAR RESTRICTED PROBLEM OF FOUR BODIES IN PRESENCE OF PERTURBATIONS IN CORIOLIS AND CENTRIFUGAL FORCES

A. AHMAD AND S. MANDAL

P. G. Department of Mathematics, Sahibganj College, Sahibganj 816 109

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In this paper the existence of the periodic orbits of first kind in the circular restricted problem of four bodies in presence of perturbations in coriolis and centrifugal forces has been established on the assumption that the three finite masses form an equilateral triangle and move in circular orbits about their centre of inertia and the infinitesimal mass moves under the gravitational field of the three finite masses without rendering the equilateral triangular configuration of the three finite masses.

1. INTRODUCTION

Bhatnagar[2] studied the existence of periodic orbits of collision in the restricted problem of four bodies. It was generalized by Bhatnagar and Taqvi[4] by considering it in a three-dimensional co-ordinate system.

Szebehely[9] considered small perturbation in coriolis force, keeping centrifugal force constant, and found that it is a stabilizing force for the triangular libration points. Bhatnagar and Hallan[3] considered small perturbations both in coriolis and centrifugal forces and found that the positions of the triangular libration points are slightly displaced from the equilateral triangular positions and form almost equilateral triangles with the primaries and for the triangular libration points the coriolis force is stabilizing and the centrifugal force is destabilizing.

These works aroused a keen interest in us to examine if the periodic orbits of first kind exist in the restricted four-body problem in presence of perturbations in coriolis and centrifugal forces. The present paper deals with this problem. The existence of the periodic orbits of first kind is established following the method used by Sharma[8].

2. Equations of Motion

Consider the motion of an infinitesimal mass P in the gravitational field of the primaries P_1, P_2, P_3 of finite masses m_1, m_2, m_3 respectively in a plane on circular orbits and let the mass of P be so small that the′ triangular configuration is not changed.

Let C be the geometric centre of the triangular configuration $P_1 P_2 P_3$ and G the centre of mass of the masses m_1, m_2, m_3 situated at P_1, P_2, P_3 respectively.

Let us take the line Gx parallel to CP_1 as x-axis and the line Gy perpendicular to Gx and in the sense of rotation as the y-axis. Let the co-ordinates of P be (x, y) and those of P_i $(i = 1, 2, 3)$ as (x_i, y_i) in the rotating system.

The presence of perturbations in coriolis and centrifugal forces may be studied with the help of parameters e_1 and e_2 respectively, where $|e_i| << 1$, $(i = 1, 2)$.

The value of the Hamiltonian H, when perturbations in coriolis and centrifugal forces are taken into consideration, is given by

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (1 + e_1)(p_1 y - p_2 x)$$

$$+ \frac{1}{2}(2e_1 - e_2)(x^2 + y^2) - \mu_1/r_1 - \mu_2/r_2 - \mu_3/r_3 \quad \text{... (1)}$$

where r_1, r_2, r_3 are the distances of P from P_1, P_2, P_3 respectively.

Now, we shift the origin to C, the geometric centre of the equilateral triangle $P_1 P_2 P_3$ through parallel axes. Let (x_1, x_2) be the co-ordinates of P and $(\bar{x}_1, \bar{x}_2)$ the co-ordinates of the centre of mass G referred to $Cx_1 x_2$ system. We have

$$(x_1, x_2) = (x + \bar{x}_1, y + \bar{x}_2).$$

In canonical form the equations of motion are

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \qquad (i = 1, 2) \quad \text{... (2)}$$

where

$$H = \frac{1}{2}(p_1^2 + p_2^2) + (1 + e_1)[p_1(x_2 - \bar{x}_2) - p_2(x_1 - \bar{x}_1)]$$

$$+ \frac{1}{2}(2e_1 - e_2)[(x_1^2 + x_2^2) - 2(x_1\bar{x}_1 + x_2\bar{x}_2) + (\bar{x}_1^2 + \bar{x}_2^2)]$$

$$- \mu_1/r_1 - \mu_2/r_2 - \mu_3/r_3 = C \quad \text{... (3)}$$

$$C = C(\mu_1, \mu_2, \mu_3)$$

$$= C_0 + \mu_1 C_1 + \mu_2 C_2 + \mu_3 C_3 + O(\mu^2).$$

3. Regularization

For the elimination of singularity at P_1, we shall use Levi-Civita's[7] parabolic transformation defined by

$$S = (A + \xi_1^2 - \xi_2^2) p_1 + 2 \xi_1 \xi_2 p_2$$

so that

$$x_i = \frac{\partial S}{\partial p_i}, \quad \pi_i = \frac{\partial S}{\partial \xi_i} \qquad (i = 1, 2)$$

where π_i are the momenta associated with the new co-ordinates ξ_i $(i = 1, 2)$ and $A = 1/\sqrt{3}$.

We also introduce a new independent variable τ instead of t given by

$$dt = r_1 \cdot d\tau, \ \tau = 0 \text{ when } t = 0.$$

The equations of motion (2) become

$$\frac{d\xi_i}{d\tau} = \frac{\partial K}{\partial \pi_i}, \quad \frac{d\pi_i}{d\tau} = -\frac{\partial K}{\partial \xi_i} \quad (i = 1, 2) \qquad \dots (4)$$

where K is the new Hamiltonian which is given by

$$\begin{aligned} K &= r_1 (H - C) \\ &= \frac{1}{8}\pi^2 + \frac{1}{2}(1 + e_1)\,[r_1 (\xi_2 \pi_1 - \xi_1 \pi_2) - A(\xi_2 \pi_1 - \xi_1 \pi_2) \\ &\quad - (\xi_1 \pi_1 + \xi_2 \pi_2)\bar{x}_2 + (\xi_2 \pi_1 + \xi_1 \pi_2)\bar{x}_1] + \frac{1}{2}(2e_1 - e_2)\, r_1 \\ &\quad \times [(A^2 + r_1^2 + 2A(\xi_1^2 - \xi_2^2)] - 2(A + \xi_1^2 - \xi_2^2)\bar{x}_1 \\ &\quad - 4\xi_1 \xi_2 \bar{x}_2 + (\bar{x}_1^2 + \bar{x}_2^2) - \mu_1 - \mu_2 r_1/r_2 - \mu_3 r_1/r_3 \\ &\quad - r_1 (C_0 + C_1 \mu_1 + C_2 \mu_2 + C_3 \mu_3) + O(\mu^2), \end{aligned} \qquad \dots (5)$$

$$\begin{aligned} r_1 &= \xi^2, \\ r_2^2 &= 1 + \xi^4 + 3A(\xi_1^2 - \xi_2^2) - 2\xi_1 \xi_2, \\ r_3^2 &= 1 + \xi^4 + 3A(\xi_1^2 - \xi_2^2) + 2\xi_1 \xi_2, \\ \xi^2 &= \xi_1^2 + \xi_2^2, \\ \pi^2 &= \pi_1^2 + \pi_2^2. \end{aligned}$$

Similar to Bhatnagar[2], we suppose that $\mu_3 = \mu$, $\mu_2 = \alpha\mu$ so that $\mu_1 = 1 - \mu(1 + \alpha)$, where α is a significant constant.

We have

$$\bar{x}_1 = (2 - 3\alpha\mu - 3\mu)A/2, \qquad \bar{x}_2 = (\alpha - 1)\mu/2.$$

The Hamiltonian (5) can be put into the form

$$K = K_0 + K_1 + \mu K_2$$

where

$$K_0 = \frac{1}{8}\pi^2 + \frac{1}{2}r_1(\xi_2\pi_1 - \xi_1\pi_2 - 2C_0') - 1, \qquad \text{... (6)}$$

$$K_1 = \frac{1}{2}e_1 r_1(\xi_2\pi_1 - \xi_1\pi_2) + \frac{1}{2}(2e_1 - e_2)r_1^3, \qquad \text{... (7)}$$

$$K_2 = -\frac{1}{4}(1 + e_1)[(\alpha - 1)(\xi_1\pi_1 - \xi_2\pi_2) + 3A(\alpha + 1)(\xi_1\pi_2 + \xi_2\pi_1)]$$

$$+ \frac{1}{2}(2e_1 - e_2)[3A(\alpha + 1)(A + \xi_1^2 - \xi_2^2) - 2(\alpha - 1)\xi_1\xi_2] - (\alpha + 1)$$

$$- \alpha r_1/r_2 - r_1/r_3 + r_1(\alpha C_1' + C_2'). \qquad \text{... (8)}$$

Now, K_0 has the same expression as in Giacaglia[6] and with him we shall assume that K_0 is negative. The solution corresponding to K_0 is given by him.

The Hamiltonian K_1 in terms of the canonical elements defined by Giacaglia[6] is written as

$$K_1 = -e_1 GD + \frac{1}{2}(2e_1 - e_2)D^3, \qquad \text{... (9)}$$

where

$$D = \frac{L^2 + (L^2 - G^2)\cos^2 l - 2L(L^2 - G^2)^{1/2}\cos l}{[-2(G + C_0)]}. \qquad \text{... (10)}$$

The canonical equations of motion for the Hamiltonian K_1 are

$$\frac{d\alpha_i}{d\tau} = \frac{\partial K_1}{\partial \beta_i} \qquad (i = 1, 2) \qquad \text{... (11)}$$

$$\frac{d\beta_i}{d\tau} = -\frac{\partial K_1}{\partial \alpha_i}$$

where

$$\alpha_1 = 1, \ \alpha_2 = g, \ \beta_1 = L, \ \beta_2 = G.$$

Equations (11) define canonical elements l, g, L, G as the functions of time, which on substitution in the solution corresponding to the Hamiltonian K_0 gives us the values of $\xi_1, \xi_2, \pi_1, \pi_2$ as functions of the canonical elements and time. This will give us the solutions corresponding to the Hamiltonian $(K_0 + K_1)$ (Sharma[8]).

The canonical equations of motion for the complete Hamiltonian K are

$$\frac{dl}{d\tau} = \frac{\partial K_0}{\partial L} + \frac{\partial K_1}{\partial L} + \mu \frac{\partial K_2}{\partial L},$$

$$\frac{dg}{d\tau} = \frac{\partial K_0}{\partial G} + \frac{\partial K_1}{\partial G} + \mu \frac{\partial K_2}{\partial G}, \qquad \text{... (12)}$$

$$\frac{dL}{d\tau} = -\frac{\partial K_1}{\partial l} - \mu \frac{\partial K_2}{\partial l},$$

$$\frac{dG}{d\tau} = -\mu \frac{\partial K_2}{\partial g},$$

where K_2 is also expressed in terms of the canonical elements l, g, L, G.

Equations (12) form the basis of the general perturbation theory for the problem under consideration.

4. Existence of Periodic Orbits of First Kind

The Hamiltonian K can be written as

$$K = (K_0 + K_1) + \mu K_2 = R_0 + \mu R_1.$$

The orbits will be periodic for $\mu \neq 0$ if the following conditions[5] are satisfied :

$$\frac{\partial [R_1]}{\partial w_i} = 0 \quad (i = 1, 2) \qquad \text{... (13)}$$

$$\frac{\partial [R_1]}{\partial a_i} = 0 \quad (i = 1, 2) \qquad \text{... (14)}$$

$$\frac{\partial (\xi_2, \eta_1, \eta_2)}{\partial (\gamma_2, \beta_1, \beta_2)} \neq 0, \quad \mu = \beta_i = \gamma_i = 0. \qquad \text{... (15)}$$

The condition (15) can be written as[1]

$$\frac{\partial^2 [R_1]}{\partial w_2^2} \cdot \Delta \neq 0 \qquad \text{... (16)}$$

where

$$\Delta = \begin{vmatrix} \dfrac{\partial^2 [R_0]}{\partial a_1^2} & \dfrac{\partial^2 [R_0]}{\partial a_1 \partial a_2} \\ \dfrac{\partial^2 [R_0]}{\partial a_1 \partial a_2} & \dfrac{\partial^2 [R_0]}{\partial a_2^2} \end{vmatrix} \qquad \text{... (17)}$$

We have

$$[R_0] = a_1[-2(a_2 + C_0)]^{1/2} - 1 - \frac{e_1 a_1 a_2}{[-2(a_2 + C_0)]^{1/2}} + \frac{(2e_1 - e_2)a_1^3}{2[-2(a_2 + C_0)]^{3/2}} . \quad ...(18)$$

Hence

$$\Delta(e_i) = [6(2e_1 - e_2)\, a_1^2 - (1 + 2e_1)\, n_l^4 - 2e_1\, a_2\, n_l^2]/n_l^6 \quad ... (19)$$

where

$$n_l = [-2(a_2 + C_0)]^{1/2} = [-2(G + C_0)]^{1/2} > 0.$$

For the classical restricted four-body problem, $e_1 = e_2 = 0$ (Bhatnagar[2]) and hence

$$\Delta(0) = -1/n_l^2 \neq 0.$$

For the present problem e_1 and e_2 are arbitrary small quantities and so when $e_i \neq 0$, $(i = 1, 2)$, we have

$$\Delta(e_i) \neq 0, \text{ provided } L^2 \neq n_l^2\, [(1 + 2e_1)\, n_l^2 + 2e_1 a_2]/[16\,(2e_1 - e_2)].$$

It can be easily seen that the condition

$$\frac{\partial [R_1]}{\partial w_2} = 0$$

is satisfied if

$$2\phi = 0,\ \pi, \text{etc. or } G = \frac{a^2}{1 + e_1}\, [1/r^3 + (1 - A)\,(2e_1 - e_2)]$$

where

$$r^2 = 1 + a + 3Aa^2.$$

Under the above conditions

$$\frac{\partial_2 [R_1]}{\partial w_2^2} = \frac{3}{2}(\alpha + 1)\, a^2\, [a/r^3 + (1 - A)\,(2e_1 - e_2)] \neq 0$$

$$\because \quad a \neq 0, \quad |e_i| \ll 1,\ (i = 1, 2)$$

Therefore, the condition (16) is satisfied for $2\phi = 0, \pi$ etc. or $G = \frac{a^2}{1 + e_1}\, [1/r^3 + (1 - A)\,(2e_1 - e_2)]$.

It can be easily verified that under these conditions for ϕ and G, the conditions (13) and (14) are also satisfied. Therefore, there exist periodic orbits of first kind of the problem under consideration.

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