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A NEW CLASS OF GENERALIZED STRONGLY NONLINEAR QUASIVARIATIONAL INEQUALITIES AND QUASICOMPLEMENTARITY PROBLEMS*

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In this paper, we study a new class of generalized strongly nonlinear quasivariational inequalities and quasicomplementarity problems. Using the projection method, we have established some existence theorems of solutions, devised new iterative algorithms for finding approximate solutions which strongly converge to the exact solution of a given problem. Our results improve and generalize many known results.

1. INTRODUCTION

Variational inequalities and complementarity problems become very effective and powerful technique for studying a wide range of problems arising in mechanics, optimization and control problems, equilibrium theory of economics, management science, operations research and other branches of mathematics and engineering sciences. In recent years, variational inequalities and complementarity problems have been extended and generalized in various directions. Quasi (implicit) variational inequalities and quasi (implicit) complementarity problems are one important extension of variational inequalities and complementarity problems. These were introduced and studied by Bensoussan, *et al.*[2] Bensoussan and Lions[3], Baiocchi and Capelo[1], Mosco[26], Dolceta[13], Pang[34, 35], Noor[28-33], Isac[19-21] and Siddiqi and Ansari[38,39]. The paper by Harker and Pang[18] provides an excellant survey on developments of the variational inequalities and complementarity problems in finite dimensional Euclidean spaces.

Another important and useful generalization of variational inequalities and complementarity problems are the generalized (quasi) variational inequalities and the generalized (quasi) complementarity problems introduced and studied by Browder[4], Rockafeller[36], Saigal[37], Fang[14, 15], Fang and Peterson[16], Siddiqi and Ansari[40], Ding[8-10], Ding and Deng[11, 12], Li and Ding[25] and Chang and Huang[6, 7].

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In this paper, we consider and study a new class of generalized strongly nonlinear quasivariational inequalities and quasicomplementarity problems. Using the projection technique, we first prove some existence theorems of solutions for the problems. Next we suggest some new iterative algorithms for finding approximate solutions and prove that the approximate solutions obtained by the iterative algorithm strongly converge to the exact solutions of these problems. Our results improve and generalize the corresponding results of Karamardian[23], Fang[15], Fang and Peterson[16], Chan and Pang[5], Ding[8-10], Noor[28-33], Siddiqi and Ansari[38-40], etc.

2. Preliminaries

Let H be a Hilbert space with its dual H^* whose norm and inner product are denoted by $\|\cdot\|$ and $(\cdot\ ,\ \cdot)$, respectively. The pairing between H^* and H is denoted by $\langle \cdot, \cdot \rangle$.

Let Λ be a canonical isomorphism from H^* onto H defined by

$$\langle u,x \rangle = (\Lambda u, x), \text{ for all } u \in H^* \text{ and } x \in H.$$

Then

$$\| \Lambda \|_{H^*} = \| \Lambda^{-1} \|_H = 1.$$

Let K be a nonempty closed convex subset of H, T, $A : H \to 2^{H^*}$ be two multivalued mappings and $g : H \to H$ be a single valued mapping. Then the problem of finding $x \in H, u \in T(x)$ and $v \in A(x)$ such that

$$g(x) \in K \text{ and } \langle u, y-g(x) \rangle \geq \langle v, y-g(x) \rangle, \text{ for all } y \in K \quad \dots (1)$$

is called generalized strongly nonlinear variational inequality problem (GSNVIP (T, A, g, K)).

If K depends on the solution x, then the problem (1) is called the generalized strongly nonlinear quasivariational inequality problem (GSNQVIP (T, A, G, $K(x)$)). More precisely, given a multivalued mapping $K : H \to 2^H$, the GSNQVIP (T, A, g, $K(x)$) is to find $x \in H, u \in T(x)$ and $v \in A(x)$ such that $g(x) \in K(x)$ and

$$\langle u, y-g(x) \rangle \geq \langle v, y-g(x) \rangle, \text{ for all } y \in K(x). \quad \dots (2)$$

Clearly, if $K(x) = K$ for all $x \in H$, the GSNQVIP (T, A, g, $K(x)$) (2) reduces to the GSNVIP(T, A, g, K)(1). In many important applications, $K(x)$ has the form

$$K(x) = m(x) + K, \text{ for each } x \in H \quad \dots (3)$$

where $m : H \to H$ is a single valued mapping.

Now, let K be a closed convex cone of H with its polar K^*, that is

$$K^* = \{y \in H^* : \langle y, x \rangle \geq 0, \text{ for all } x \in K\}.$$

Related to the GSNQVIP(T, A, g, $K(x)$), we consider and study a new class of complementarity problem which intends to find $x \in H$, $u \in T(x)$ and $v \in A(x)$ such that $g(x) \in K(x)$ and

$$\langle u - v, g(x) - m(x) \rangle = 0, \qquad \dots (4)$$

where $K(x)$ is given by (3) and $K^*(x)$ is the polar cone of $K(x)$, i.e.

$$K^*(x) = \{y \in H^* : \langle y, z \rangle \geq 0 \text{ for all } z \in K(x)\}.$$

This type of problem is called a generalized strongly nonlinear quasicomplementarity problem (GSNQCP(T, A, g, $K(x)$)).

Special cases — (a) If T and A are single valued mappings, then the problem (2) reduces to the GSNQVIP(T, A, g, $K(x)$) considered by Siddiqi and Ansari[39].

(b) If $m(x) = 0$ for all $x \in H$, then the problem (2) reduces to the problem (1) which is considered and studied by Ding and Deng[12].

(c) If $m(x) = 0$ for all $x \in H$ and T and A are single valued mappings, then the problem (2) reduces to the improving version of the problem (2.1) considered by Noor[33] and the problem (4) reduces to the problem (2.5) of Noor[33].

(d) If $g = I$, the identity mapping, then the problem (2) reduces to the problem (2.2) considered by Ding[9] and the problem (4) reduces to the problem (2.3) of Ding[9].

In brief, the problems (2) and (4) are the most general and unifying forms of various extended classes of variational inequalities and complementarity problems. As special cases, they include the following problems considered by Ding[8], Ding and Deng[11], Li and Ding[25], Saigal[37], Fang and Peterson[16], Noor[28-33], Isac[19-21], Siddiqi and Ansari[38-40], Chang and Huang[6, 7] and others.

Let K be a closed convex subset of a Hilbert space H. We recall that if P_K denotes the projection onto K, that is, for each $x \in H$, $P_K(x)$ is the unique element satisfying

$$\| x - P_K(x) \| = \min_{y \in K} \| x - y \|.$$

We need the following known results.

Lemma 2.1[24] — Let K be a nonempty convex subset of H. Then, given $z \in H$, we have $x = P_K(z)$ if and only if

$$(x - z,\ y - x) \geq 0, \text{ for all } y \in K.$$

Lemma 2.2[24] — $P_K : H \to K$ is nonexpansive, i.e.,

$$\| P_K(x) - P_K(y) \| \leq \| x - y \|, \text{ for all } x, y \in H.$$

Lemma 2.3[29] — Let K be a nonempty closed convex subset of H, $m : H \to H$ and $K(x) = m(x) + K$ for each $x \in H$. Then for all $x, y \in H$,

$$P_{K(x)}(y) = m(x) + P_K(y - m(x)).$$

Lemma 2.4[7] — Let $m : H \to H$ and $K(x) = m(x) + K$ for each $x \in H$. Then $K^*(x) = (m(x) + K)^* = m^*(x) \cap K^*$ where $m^*(x) = \{y \in H^* : \langle y, m(x) \rangle \geq 0\}$.

Definition 2.1 — A mapping $g : H \to H$ is said to be

(i) γ-strongly monotone if there exists a constant $\gamma > 0$ such that

$$(g(x) - g(y), x - y) \geq \gamma \|x-y\|^2, \text{ for all } x, y \in H;$$

(ii) σ-Lipschitz continuous if there exists a constant $\sigma \geq 0$ such that

$$\|g(x) - g(y)\| \leq \sigma \|x - y\|, \text{ for all } x, y \in H.$$

Definition 2.2 — A set-valued mapping $T : H \to 2^{H^*}$ is said to be

(i) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle u - v, x - y \rangle \geq \alpha \|x-y\|^2, \text{ for all } x, y \in H, u \in T(x) \text{ and } v \in T(y);$$

(ii) β-Lipschitz continuous if there exists a constant $\beta \geq 0$ such that

$$\delta(T(x), T(y)) \leq \beta \|x - y\|, \text{ for all } x, y \in H$$

where $\delta(A, B) = \sup\{\|a - b\| : a \in A, b \in B\}$ for any $A, B \in 2^{H^*}$.

Definition 2.3 — Let $g : H \to H$. A set-valued mapping $T : H \to C(H^*)$ is said to be (i) α-strongly monotone with respect to g if there exist a constant $\alpha > 0$ such that for all $x, y \in H$, $u \in T(x)$ and $v \in T(y)$,

$$\langle u - v, g(x) - g(y) \rangle \geq \alpha \|g(x) - g(y)\|^2;$$

(ii) β-H-Lipschitz continuous with respect to g if there exists a constant $\beta \geq 0$ such that for all $x, y \in H$,

$$H(T(x), T(y)) \leq \beta \|g(x) - g(y)\|$$

where $C(H^*)$ is the family of all nonempty compact subsets of H^* and $H(\cdot\,,\cdot)$ is the Hausdorff metric on $C(H^*)$.

3. Main Results

In this section, we first prove that the GSNQVIP(T, A, g, $K(x)$) and the GSNQCP(T, A, g, $K(x)$) have the same set of solutions if K is a closed convex cone of H and $K(x)$ has the form (3). Next, we prove some existence theorems of solutions for the GSNQVIP(T, A, g, $K(x)$) and the GSNQCP(T, A, g, $K(x)$) and suggest some new iterative algorithms for finding approximate solutions. Finally, we show that the approximate solutions converge to the exact solution of these problems.

Theorem 3.1 — Let K be a closed convex cone of H. $T, A : H \to 2^{H^*}$, $g, m : H \to H$ and $K(x) = m(x) + K$ for all $x \in X$. Then the GSNQVIP(T, A, g, $K(x)$) and the GSNQCP(T, A, g, $K(x)$) have the same set of solutions.

PROOF : Let (x, u, v) is a solution of the GSNQVIP$(T, A, g, K(x))$ (2), then $x \in H$, $u \in T(x)$ and $v \in A(x)$ such that $g(x) \in K(x)$ and

$$\langle u, y - g(x) \rangle \geq \langle v, y - g(x) \rangle, \text{ for all } y \in K(x). \qquad \text{... (5)}$$

Hence $g(x) - m(x) \in K$ which implies $2(g(x) - m(x)) \in K$ and $2g(x) - m(x) \in K(x)$. By (5), we have

$$\langle u, g(x) - m(x) \rangle \geq \langle v, g(x) - m(x) \rangle.$$

Since $0 \in K$, we have $m(x) \in K(x)$. It follows from (5) that

$$\langle u, g(x) - m(x) \rangle \leq \langle v, g(x) - m(x) \rangle.$$

Hence we must have

$$\langle u - v, g(x) - m(x) \rangle = 0. \qquad \text{... (6)}$$

For any $z \in K(x)$, since $m(x) \in K(x)$ and $K(x)$ is also cone, we have $m(x) + z \in K(x)$. It follows from (5) and (6) that

$$\langle u - v, z \rangle = \langle u - v, z \rangle + \langle u - v, m(x) - g(x) \rangle$$
$$= \langle u - v, m(x) + z - g(x) \rangle \geq 0$$

and hence $u - v \in K^*(x)$. This proves that (x, u, v) is a solution of the GSNQCP$(T, A, g, K(x))$.

Conversely, suppose that (x, u, v) is a solution of the GSNQCP$(T, A, g, K(x))$, then $u - v \in K^*(x)$ and $\langle u - v, g(x) - m(x) \rangle = 0$. For any $y \in K(x)$, $y = m(x) + z$ for some $z \in K$. It follows from $u - v \in K^*(x) = m^*(x) \cap K^*$ that

$$\langle u - v, y - g(x) \rangle = \langle u - v, m(x) + z - g(x) \rangle$$
$$= \langle u - v, m(x) - g(x) \rangle + \langle u - v, z \rangle$$
$$= \langle u - v, z \rangle \geq 0,$$

i.e., (x, u, v) is a solution of the GSNQVIP$(T, A, g, K(x))$. This completes the proof.

Remark 3.1 : Theorem 3.1 generalizes Lemma 3.1 of Noor[29, 30, 32], Proposition 2.1 of Chan and Pang[5], Theorem 3.1 of Ding[9] and the earlier results of Karamardian[23] and Fang[14].

Theorem 3.2 — Let K be a nonempty closed convex subset of H and $K(x)$ has the form (3). Then the GSNQVIP$(T, A, g, K(x))$ has a solution (x^*, u^*, v^*) if and only if x^* is a fixed point of the mapping $F : H \to 2^H$ defined by

$$F(x) = \bigcup_{u \in T(x)} \bigcup_{u \in A(x)} [x - g(x) + m(x) + P_K(g(x) - \rho\Lambda(u - v) - m(x))]$$

for each $x \in H$ where ρ is some positive constant.

PROOF : It is similar to the proof of Theorem 3.2 of Ding[9].

Remark 3.2 : Theorem 3.2 generalizes Theorem 3.2 of Ding[9], Lemma 3.1 of Siddiqi and Ansari[38, 39] and Lemma 3.1 of Noor[33].

Now, we consider the iterative algorithms for finding approximate solutions of the GSNQVIP(T, A, g, $K(x)$).

Algorithm 3.1 — Let K be a nonempty closed convex subset of H and $K(x)$ has the form (3) where $m : H \to H$. Let $g : H \to H$ be such that $K(x) \subset g(H)$ for all $x \in H$ and $T, A : H \to C(H^*)$. For any given $x_0 \in H$, choose $u_0 \in T(x_0)$ and $v_0 \in A(x_0)$ and let

$$w_0 = m(x_0) + P_K(g(x_0) - \rho\Lambda(u_0 - v_0) - m(x_0)) \in K(x_0) \subset g(H).$$

Hence there exists $x_1 \in H$ such that $w_0 = g(x_1)$. Since $u_0 \in T(x_0) \in C(H^*)$ and $v_0 \in A(x_0) \in C(H^*)$, by Nadler[27], there exist $u_1 \in T(x_1)$ and $v_1 \in A(x_1)$ such that

$$\| u_0 - u_1 \| \le H(T(x_0), T(x_1)) \text{ and } \| v_0 - v_1 \| \le H(A(x_0), A(x_1)).$$

Let

$$w_1 = m(x_1) + P_K(g(x_1) - \rho\Lambda(u_1 - v_1) - m(x_1) \in K(x_1) \subset g(H).$$

Thus there exists $x_2 \in H$ such that $w_1 = g(x_2)$. By induction, we can obtain iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ as follows : for $n \ge 0$,

$$u_n \in T(x_n), \quad v_n \in A(x_n),$$

$$\| u_{n+1} - u_n \| \le H(T(x_{n+1}), T(x_n)),$$

$$\| v_{n+1} - v_n \| \le H(A(x_{n+1}), A(x_n)),$$

$$g(x_{n+1}) = m(x_n) + P_K(g(x_n) - \rho\Lambda(u_n - v_n) - m(x_n)),$$

where $\rho > 0$ is a constant.

Several special cases of Algorithm 3.1 are listed below :

(i) If $g(x) = x$ for all $x \in H$ and T, A are single-valued mappings, Algorithm 3.1 reduces to the Algorithm 3.1 of Siddiqi and Ansari[39].

(ii) If $m(x) = 0$ for all $x \in H$, i.e., $K(x) = K$ and T is a single-valued mapping, Algorithm 3.1 reduces to the Algorithm 2.1 of Chang and Huang[6].

Algorithm 3.2 — Let K be a nonempty closed convex subset of H and $K(x)$ have the form (3) where $m : H \to H$. Let $g : H \to H$ and $T, A : H \to 2^{H^*}$. Given $x_0 \in H$, the iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ are defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n [y_n - g(y_n) + m(y_n)$$
$$+ P_K(g(y_n) - \rho\Lambda(u_n - v_n) - m(y_n))],$$

$$y_n = (1 - \beta_n) x_n + \beta_n [x_n - g(x_n) + m(x_n)$$
$$+ P_K(g(x_n) - \rho\Lambda(\overline{u}_n - \overline{v}_n) - m(x_n))]$$

for $n \geq 0$ where $u_n \in T(y_n)$, $v_n \in A(y_n)$, $\bar{u}_n \in T(x_n)$, $\bar{v}_n \in A(x_n)$, $0 \leq \alpha_n$, $\beta_n \leq 1$, $\Sigma_{n=0}^{\infty}$ α_n diverges and $\rho > 0$ is a constant.

Several special cases of Algorithm 3.2 are listed below :

(i) If $m(x) = 0$ for all $x \in H$, Algorithm 3.2 reduces to the Algorithm 3.1 of Ding and Deng[11].

(ii) If $m(x) = 0$ and $g(x) = x$ for all $x \in H$ and $\beta_n = 0$ for all $n \geq 0$, Algorithm 3.2 reduces to the Algorithm in Theorem 3.4 of Ding[9].

(iii) If T and A are single-valued mappings and $\beta_n = 0$, $\alpha_n = 1$ for all $n \geq 0$, Algorithm 3.2 reduces to the Algorithm 3.1 of Siddiqi and Ansari[40].

(iv) If $m(x) = 0$ for all $x \in H$, T and A are single-valued mappings and $\beta_n = 0$, $\alpha_n = 1$ for all $n \geq 0$, Algorithm 3.2 reduces to the Algorithm 3.1 of Noor[33].

Now, we prove some existence theorems of solutions for the GSNQVIP(T, A, g, $K(x)$).

Theorem 3.3 — Let K be a nonempty closed convex subset of H and $K(x)$ has the form (3) where $m : H \to H$. Let $g : H \to H$ be such that $K(x) \subset g(H)$ for all $x \in H$ and $g(H)$ is closed. Suppose that $T : H \to C(H^*)$ is α-strongly monotone β-H-Lipschitz continuous with respect to g, $A : H \to C(H^*)$ is γ-H-Lipschitz continuous with respect to g, and m is μ-Lipschitz continuous with respect to g. If there exists a constant $\rho > 0$ such that

$$\rho\gamma + 2\mu < 1, \quad \gamma < \alpha \leq \beta,$$

$$\left| \rho - \frac{\alpha - \gamma(1-2\mu)}{\beta^2 - \gamma^2} \right| \leq \frac{\sqrt{[\alpha - \gamma(1-2\mu)]^2 - 4(\beta^2 - \gamma^2)\,\mu(1-\mu)}}{\beta^2 - \gamma^2}, \qquad \dots (7)$$

then the iterative sequences $\{g(x_n)\}$, $\{u_n\}$ and $\{v_n\}$ defined by the Algorithm 3.1 strongly converge to $g(x^*)$, u^* and v^*, respectively and (x^*, u^*, v^*) is a solution of the GSNQVIP(T, A, g, $K(x)$).

PROOF : By the Algorithm 3.1 and Lemma 2.2, we have

$$\begin{aligned}
&\| g(x_{n+1}) - g(x_n) \| \\
&= \| m(x_n) + P_K(g(x_n) - \rho\Lambda(u_n - v_n) - m(x_n)) \\
&\qquad - m(x_{n-1}) - P_K(g(x_{n-1}) - \rho\Lambda(u_{n-1} - v_{n-1}) - m(x_{n-1})) \| \\
&\leq 2\| m(x_n) - m(x_{n-1}) \| + \| g(x_n) - g(x_{n-1}) - \rho\Lambda(u_n - u_{n-1}) \| \\
&\qquad + \rho \| v_n - v_{n-1} \| \\
&\leq 2\| m(x_n) - m(x_{n-1}) \| + \| g(x_n) - g(x_{n-1}) - \rho\Lambda(u_n - u_{n-1}) \| \\
&\qquad + \rho H(A(x_n), A(x_{n-1})).
\end{aligned}$$

Since T is α-strongly monotone β-H-Lipschitz continuous with respect to g,

$$\begin{aligned}
&\| g(x_n) - g(x_{n-1}) - \rho\Lambda\,(u_n - u_{n-1}) \|^2 \\
&\quad = \| g(x_n) - g(x_{n-1}) \|^2 - 2\rho \langle u_n - u_{n-1}, g(x_n) - g(x_{n-1}) \rangle \\
&\qquad\qquad + \rho^2 \beta^2 \| g(x_n) - g(x_{n-1}) \|^2 \\
&\quad \le (1 - 2\rho\alpha + \rho^2 \beta^2)\; \| g(x_n) - g(x_{n-1}) \|^2.
\end{aligned}$$

Note that m is μ-Lipschitz continuous with respect to g and A is γ-H-Lipschitz continuous with respect to g, it follows that

$$\begin{aligned}
&\| g(x_{n+1}) - g(x_n) \| \\
&\quad \le (2\mu + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\gamma)\; \| g(x_n) - g(x_{n-1}) \| \\
&\quad = \theta \| g(x_n) - g(x_{n-1}) \| \qquad \text{... (8)}
\end{aligned}$$

where $\theta = 2\mu + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2} + \rho\gamma < 1$ by the condition (7). It follows that $\{g(x_n)\}$ is a Cauchy sequence in $g(H)$. Since $g(H)$ is closed, there exists $x^* \in H$ such that $\{g(x_n)\}$ strongly converge to $g(x^*)$. Since T is β-H-Lipschitz continuous and A is γ-H-Lipschitz continuous with respect to g, it follows from the Algorithm 3.1 that

$$\| u_{n+1} - u_n \| \le H(T(x_{n+1}), T(x_n)) \le \beta \| g(x_{n+1}) - g(x_n) \|,$$

$$\| v_{n+1} - v_n \| \le H(A(x_{n+1}), A(x_n)) \le \gamma \; \| g(x_{n+1}) - g(x_n) \|.$$

Hence $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in H^*. Let $\{u_n\}$ and $\{v_n\}$ strongly converge to u^* and v^*, respectively. Define

$$w^* = m(x^*) + P_K(g(x^*) - \rho\Lambda(u^* - v^*) - m(x^*)) \in K(x^*)$$

we have

$$\begin{aligned}
&\| g(x_{n+1}) - w^* \| \\
&\quad = \| m(x_n) + P_K(g(x_n) - \rho\Lambda\,(u_n - v_n) - m(x_n)) \\
&\qquad\qquad - m(x^*) - P_K(g(x^*) - \rho\Lambda\,(u^* - v^*) - m(x^*)) \| \\
&\quad \le 2 \| m(x_n) - m(x^*) \| + \| g(x_n) - g(x^*) \| \\
&\qquad\qquad + \rho \| u_n - v_n \| + \rho \| v_n - v^* \| \\
&\quad \le (1 + 2\mu) \| g(x_n) - g(x^*) \| + \rho(\| u_n - u^* \| + \| v_n - v^* \|) \to 0,
\end{aligned}$$

and hence

$$g(x^*) = w^* = m(x^*) + P_K(g(x^*) - \rho\Lambda\,(u^*) - v^*) - m(x^*)) \in K(x^*).$$

On the other hand, we have

$$d(u^*, T(x^*)) \le \| u^* - u_n \| + d(u_n, T(x^*))$$
$$\le \| u^* - u_n \| + H(T(x_n), T(x^*))$$
$$\le \| u^* - u_n \| + \beta \| g(x_n) - g(x^*) \| \to 0$$

and so $u^* \in T(x^*)$. By a similar argument, we have $v^* \in A(x^*)$. Hence

$$x^* \in F(x^*) = \bigcup_{u \in T(x^*)} \bigcup_{v \in A(x^*)} [x^* - g(x^*) + m(x^*) + P_K(g(x^*) - \rho\Lambda(u^* - v^*) - m(x^*))].$$

It follows from Theorem 3.2 that (x^*, u^*, v^*) is a solution of the GSNQVIP(T, A, g, $K(x)$). This completes the proof.

Remark 3.3 : If $m(x) = 0$ and $g(x) = x$ for all $x \in H$, then Theorem 3.3 improves the corresponding results of Noor[28] and Glowinski *et al.*[17].

Theorem 3.4 — Let K be a nonempty closed convex cone of H. $K(x)$, m, g, T, A satisfy the assumptions in Theorem 3.3 and the condition (7) holds. Then the iterative sequences $\{g(x_n)\}$, $\{u_n\}$, $\{v_n\}$ defined by the Algorithm 3.1 strongly converge to $g(x^*)$, u^*, v^*, respectively and (x^*, u^*, v^*) is a solution of the GSNQCP(T, A, g, $K(x)$).

PROOF : The conclusion holds from Theorem 3.3 and Theorem 3.1.

Remark 3.4 : If $m(x) = 0$ for all $x \in H$ and T is a single-valued mapping, then Theorem 3.4 reduces to Theorem 3.1 of Chang and Huang[6]. Theorem 3.4 improves and generalizes Theorem 3.2 of Noor[31] in many ways.

ieorem 3.5 — Let K be a nonempty closed convex subset of H. $K(x)$ has the form (3), $T : H \to 2^{H^*}$ is α-strongly monotone and β-Lipschitz continuous, $A : H \to 2^{H^*}$ is γ-Lipschitz continuous, $g : H \to H$ is λ-strongly monotone and σ-Lipschitz continuous and $m : H \to H$ is μ-Lipschitz continuous. Suppose there exists a constant $\rho > 0$ such that

$$k = 2(\mu + \sqrt{1 - 2\lambda + \sigma^2}) < 1, \ \alpha > \gamma(1 - k) + \sqrt{(\beta^2 - \gamma^2)k(2 - k)},$$
$$\left| \rho - \frac{\alpha + \gamma(k-1)}{\beta^2 - \gamma^2} \right| < \frac{\sqrt{(\alpha + \gamma(k-1))^2 - (\beta^2 - \gamma^2)k(2-k)}}{\beta^2 - \gamma^2}. \quad \dots (9)$$

Then the GSNQVIP(T, A, g, $K(x)$) has a solution (x^*, u^*, v^*) and the iterativ sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ defined by the Algorithm 3.2 strongly converge to x^*, u^* and v^*, respectively.

PROOF : We first prove that the GSNQVIP(T, A, g, $K(x)$) has a solution (x^*, u^*, v^*). By Theorem 3.2, it is suffices to prove that the mapping F defined in Theorem 3.2 has a fixed point x^* in H. For any $x, y \in H$, $a \in F(x)$ and $b \in F(y)$, there exist $u_1 \in T(x)$, $v_1 \in A(x)$, $u_2 \in T(y)$ and $v_2 \in A(y)$ such that

$$a = x - g(x) + m(x) + P_K(g(x) - \rho\Lambda(u_1 - v_1) - m(x)),$$

$$b = y - g(y) + m(y) + P_K(g(y) - \rho\Lambda(u_2 - v_2) - m(y)).$$

By Lemma 2.2, we have

$$\begin{aligned}\|a - b\| &\le \|x - y - (g(x) - g(y))\| + \|m(x) - m(y)\| \\ &\quad + \|P_K(g(x) - \rho\Lambda(u_1 - v_1) - m(x)) \\ &\quad - P_K(g(y) - \rho\Lambda(u_2 - v_2) - m(y))\| \\ &\le 2\|x - y - (g(x) - g(y))\| + 2\|m(x) - m(y)\| \\ &\quad + \|x - y - \rho\Lambda(u_1 - u_2)\| + \rho\delta(A(x), A(y)).\end{aligned}$$

Since T and g are both strongly monotone and Lipschitz continuous and m is Lipschitz continuous, by using the technique of Noor[29], we have

$$\|x - y - (g(x) - g(y))\| \le \sqrt{1 - 2\lambda + \sigma^2}\,\|x - y\|,$$

$$\|x - y - \rho\Lambda(u_1 - u_2)\| \le \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\,\|x - y\|,$$

$$\delta(A(x), A(y)) \le \gamma\|x - y\|,$$

$$\|m(x) - m(y)\| \le \mu\|x - y\|.$$

It follows that

$$\begin{aligned}\delta(F(x), F(y)) &\le [2(\mu + \sqrt{1 - 2\lambda + \sigma^2}) + \sqrt{1 - 2\alpha\rho + \rho^2\beta^2} + \rho\gamma]\,\|x - y\| \\ &= [k + t(\rho) + \rho\gamma]\,\|x - y\| = \theta\|x - y\|, \qquad \text{... (10)}\end{aligned}$$

where $t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}$ and $\theta = k + t(\rho) + \rho\gamma$. By the condition (9), we have $\theta < 1$. It follows from (10) and Theorem 3.1 of Siddiqi and Ansari[40] that F has a fixed point $x^* \in H$. By Theorem 3.2, there exist $u^* \in T(x^*)$ and $v^* \in A(x^*)$ such that (x^*, u^*, v^*) is a solution of the GSNQVIP(T, A, g, $K(x)$).

Next, we prove that the iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ strongly converge to x^*, u^* and v^*, respectively. Since (x^*, u^*, v^*) is a solution of the GSNQVIP(T, A, g, $K(x)$), we have

$$x^* = x^* - g(x^*) + m(x^*) + P_K(g(x^*) - \rho\Lambda(u^* - v^*) - m(x^*)).$$

By using the similar argument as above, we obtain

$$\|x_n - x^* - (g(x_n) - g(x^*))\| \le \sqrt{1 - 2\lambda + \sigma^2}\,\|x_n - x^*\|,$$

$$\|x_n - x^* - \rho\Lambda(\bar{u}_n - u^*)\| \le \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}\,\|x_n - x^*\|,$$

$$\| y_n - x^* - (g(y_n) - g(x^*)) \| \le \sqrt{1 - 2\lambda + \sigma^2} \, \| y_n - x^* \|,$$

$$\| y_n - \rho A(u_n - u^*) \| \le \sqrt{1 - 2\alpha\rho + \rho^2 \beta^2} \, \| y_n - x^* \|.$$

Thus, by Lemma 2.2 and the Lipschitz continuity of m and A, we obtain

$$\begin{aligned}
\| y_n - x^* \| &= \| (1 - \beta_n) x_n + \beta_n [x_n - g(x_n) + m(x_n) + P_K (g(x_n) \\
&\quad - \rho A(\bar{u}_n - \bar{v}_n) - m(x_n))] - (1 - \beta_n) x^* \\
&\quad - \beta_n [x^* - g(x^*) + m(x^*) + P_K (g(x^*) \\
&\quad - \rho A(u^* - v^*) - m(x^*))] \| \\
&\le (1 - \beta_n) \| x_n - x^* \| + 2\beta_n \| m(x_n) - m(x^*) \| \\
&\quad + 2\beta_n \| x_n - x^* - (g(x_n) - g(x^*)) \| \\
&\quad + \beta_n \| x_n - x^* - \rho A (\bar{u}_n - u^*) \| + \beta_n \rho \delta (A(x_n), A(x^*)) \\
&\le (1 - \beta_n) \| x_n - x^* \| + \beta_n k \| x_n - x^* \| \\
&\quad + \beta_n t(\rho) \| x_n - x^* \| + \beta_n \rho\gamma \| x_n - x^* \| \\
&\le (1 - \beta_n) \| x_n - x^* \| + \beta_n \theta \| x_n - x^* \| \\
&\le \| x_n - x^* \|. \qquad \text{... (11)}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\| x_{n+1} - x^* \| &= \| (1 - \alpha_n) x_n + \alpha_n [y_n - g(y_n) + m(y_n) + P_K (g(y_n) \\
&\quad - \rho A(u_n - v_n) - m(y_n))] - (1 - \alpha_n) x^* - \alpha_n [x^* - g(x^*) \\
&\quad + m(x^*) + P_K (g(x^*) - \rho A (u^* - v^*) - m(x^*))] \| \\
&\le (1 - \alpha_n) \| x_n - x^* \| + \alpha_n \theta \| y_n - x^* \|. \qquad \text{... (12)}
\end{aligned}$$

It follows from (11) and (12) that

$$\begin{aligned}
\| x_{n+1} - x^* \| &\le (1 - \alpha_n) \| x_n - x^* \| + \alpha_n \theta \| x_n - x^* \| \\
&= [1 - (1 - \theta) \alpha_n] \| x_n - x^* \| \\
&\le \prod_{i=0}^{n} [1 - (1 - \theta) \alpha_i] \| x_0 - x^* \|.
\end{aligned}$$

Since $\Sigma_{n=0}^{\infty} \alpha_n$ diverges and $1 - \theta > 0$, we have $\Pi_{i=0}^{\infty} [1 - (1 - \theta) \alpha_i] = 0$. Hence, the sequence $\{x_n\}$ strongly converges to x^*. By (11), the sequence $\{y_n\}$ also strongly converges to x^*. Since $u_n \in T(y_n)$, $u^* \in T(x^*)$ and T is β-Lipschitz continuous, we have

$$\| u_n - u^* \| \leq \delta\,(T(y_n), T(x^*)) \leq \beta \, \| y_n - x^* \| \to 0,$$

and hence the sequence $\{u_n\}$ strongly converges to u^*. Similarly, we can show that the sequence $\{v_n\}$ strongly converges to v^*. This completes the proof.

Remark 3.5 : If $m(x) = 0$ for all $x \in H$, Theorem 3.5 reduces to Theorem 3.1 of Ding and Deng[11]; if $g(x) = x$ for all $x \in H$ and $\beta_n = 0$ for all $n \geq 0$, Theorem 3.5 improves Theorem 3.5 of Ding[9]; if T and A are single-valued mappings and $\beta_n = 0$, $\alpha_n = 1$ for all $n \geq 0$, Theorem 3.5 improves Theorem 3.1 of Siddiqi and Ansari[40]; if T and A are single-valued mappings, $m(x) = 0$ for all $x \in H$ and $\beta_n = 0$, $\alpha_n = 1$ for all $n \geq 0$, Theorem 3.5 improves Theorem 3.1 of Noor[33]. We emphasize that, in Theorem 3.5, we not only prove the existence of solutions but also give a new iterative algorithm approximating solutions.

Theorem 3.6 — Let K be a closed convex cone of H. $K(x)$, m, g, T, A satisfy all assumptions in Theorem 3.5 and the condition (9) holds. Then the GSNQCP(T, A, g, $K(x)$) has a solution (x^*, u^*, v^*) and the iterative sequences $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ defined by the Algorithm 3.2 strongly converge to x^*, u^*, v^*, respectively.

PROOF : The conclusion follows from Theorem 3.5 and Theorem 3.1.

Remark 3.6 : If $A = 0$, T is a single-valued mapping and $\beta_n = 0$, $\alpha_n = 1$ for all $n \geq 0$, Theorem 3.6 improves Theorem 3.1 of Noor[32]. Theorem 3.6, in turn, generalizes the responding results of Noor[30-32] and Fang[15].

4. CONCLUDING REMARKS

Our main aim in this study has been to introduce a new class of generalized strongly nonlinear quasivariational inequalities and quasicomplementarity problems which include the most of known variational inequalities and complementarity problems in the literature as special cases. We first proved the generalized strongly nonlinear quasivariational inequality and the generalized strongly nonlinear quasicomplementarity problem have the same set of solutions. Then, by applying the projection technique and fixed point theorem, we obtained some existence theorems of solutions for these problems. Next, following Ishikawa[22] type iterative algorithm in fixed point theory, we suggested some new iterative algorithms for finding approximate solutions and proved that the approximate solutions given by the new iterative algorithms strongly converge to the exact solutions of these problems under some assumptions. Our iterative algorithms include the most of known iterative algorithms in recent literature as special cases.

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