Indian J. pure appl. Math., 25(11): 1135-1146, November 1994

PERIODIC BOUNDARY VALUE PROBLEMS FOR SECOND ORDER NONLINEAR COUPLED ORDINARY DIFFERENTIAL SYSTEMS

K. NARSIMHA REDDY*

Department of Mathematics, Osmania University, Hyderabad 500 007

(*Received* 3 *September* 1993; *accepted* 20 *June* 1994)

This paper is concerned with the results on existence of solutions of coupled nonlinear second order ordinary differential systems subject to periodic boundary conditions.

1. INTRODUCTION

As many phenomena in nature are either periodic or nearly periodic, it is appropriate to study the existence of periodic solutions of differential or integral equations. A large literature on the existence of solutions for periodic boundary value problems (PBVP's for short) of nonlinear differential equations is available, for example, see under references [9]-[13] and the references given there. For a general treatment of boundary value problems we refer the readers to Bernfeld and Lakshmikantham[1]. Most of these works are concerned with the existence of solutions for PBVP's by the composition of two basic techniques, namely the method of upper and lower solutions and the alternative method. The work of Gupta[8] is concerned with the problem of existence of solutions to periodic boundary value problems for coupled first order differential equations. In this paper we extend the results of Gupta[8] and others to the corresponding second order coupled systems.

The organization of the present work is as follows : Our notation and terminology are fairly consistent and can be understood by referring to earlier works[2,4,5,7,8]. However for the sake of completeness we describe them briefly and further state the basic result required for our subsequent discussion in section 2. Section 3 deals with the main existence results and their consequences.

2. PRELIMINARIES AND BASIC RESULTS

Let X and Y be two Banach spaces with a bilinear pairing to the reals denoted by (x, y), $x \in X$, $y \in Y$ such that $|(x,y)| \leq \|x\|_X \|y\|_Y$. Consider the operator equation

*Present address : Department of Mathematics, Osmania University, Post Graduate Centre, Nalgonda 508 001.

$$Lu + Nu = W \qquad \text{... (2.1)}$$

in which $L : D(L)\ X \to Y$ is a linear Fredholm mapping and $N : X \to Y$, a nonlinear mapping. Now we shall employ a method which is sharper than the wellknown alternative method, see Breziz[2], Gupta[7] and Calvert and Gupta[5]. Similar results for (2.1) have been obtained by Mawhin and others via the use of the theory of coincidence degree, see for example Gaines and Mawhin[6] and the references therein.

Suppose that $X \subset Y$ continuously. Let $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$ where $X_2 =$ Ker L, $Y_1 = R(L)$, $X_1 \subset Y_1$ and $X_2 \subset Y_2$. Further suppose that for $x = x_1 + x_2 \in X$, $x_1 \in X_1$, $x_2 \in X_2$, $y = y_1 + y_2 \in Y$, $y_1 \in Y_1$, $y_2 \in Y_2$, we have

$$(x, y) = (x_1, y_1) + (x_2, y_2).$$

Let $P : Y \to Y_1$ and $Q : Y \to Y_2$ be continuous projection mappings such that $P(X) \subseteq X_1$, $Q(X) \subseteq X_2$ and $P/X : X \to X_1$, $Q/X : X \to X_2$ are continuous projections. Suppose that there exists a bounded linear mapping $K : Y_1 \to X_1$ such that for $u \in Y$, $KP(u) \in D(L)$, $LKP(u) = P(u)$ and $(KP(u), P(u)) \geq 0$. Let $N : X \to X \subset Y$ be a mapping such that $KPN : X \to X_1$ is compact. The following is the basic result, Theorem 1.1 of Gupta[8].

Theorem 2.1 — Let $L : D(L) \subset X \to Y$ be a linear Fredholm mapping $N : X \to X \subset Y$ be a continuous mapping such that KPN is compact. Further suppose that

(H_1) there exist constants $a > 0$, $b > 0$ such that

$$(Nu, u) \geq a \| Qu \|_Y - b \ \text{ for } u \in X \qquad \text{... (2.2)}$$

(H_2) there exists a constant $\alpha \geq 0$ and for every $k > 0$ there corresponds a constant $c(k)$ such that

$$(Nu, u) \geq k \| Nu \|_Y - \alpha \| u \|_X - c(k), \ \text{ for } u \in X. \qquad \text{... (2.3)}$$

Then for each $W_1 \in Y_1$, the equation $Lu + Nu = W_1$ has atleast one solution u in X.

Definition 2.2 — A mapping $N : Y^* \to Y$ is said to be monotone if $(NU - Nv, U - v) \geq 0$ for $u, v \in Y^*$, the dual of the space Y. Further N is said to be trimonotone if for $u, v, W \in Y^*$, $(Nu, u - v) + (Nv, v - W) + (NW, W - u) \geq 0$. Observe that a trimonotone mapping is monotone while the converse is not true in general.

We notice that hypothesis (H_2) of Theorem 2.1 may be verified in the case when the operator N is monotone with the help of the following useful lemma (Brezis and Browder[3, 4]).

Lemma 2.3 — Let $N : Y^* \to Y$ be a bounded mapping satisfying the following condition :

There exists a function $c : Y^* \times Y^* \to R$ which is bounded on bounded subsets of $Y^* \times Y^*$ such that for $u, v, W \in Y^*$,

$$(Nu, u - v) + (Nv, v - W) + (NW, W - u) \geq -c(v, W). \qquad \text{... (2.4)}$$

Then for every $k > 0$, there exists a constant $c(k)$ such that

$$(Nu, u) \geq k \| Nu \|_Y - \| NO \|_Y \| u \|_{Y^*} - c(k), \text{ for } u \in Y^*. \quad \text{... (2.5)}$$

3. Results on Existence for Coupled Systems

In this section we discuss the existence of solutions of the system

$$\begin{aligned} -u'' &= f(t, u, v) + p(t), \\ -v'' &= g(t, u, v) + g(t), \quad t \in [0, T] \end{aligned} \quad \text{... (3.1)}$$

satisfying the boundary conditions

$$\begin{aligned} u(0) &= u(T), u'(0) = u'(T) \\ v(0) &= v(T), v'(0) = v'(T) \end{aligned} \quad \text{... (3.2)}$$

in which $f, g \in C([0, T] \times R^n \times R^n, R^n)$ and $p, q \in C([0, T], R^n)$.

Furthermore $\int_0^T p(t)\, dt = \int_0^T q(t)\, dt = 0.$

We introduce the following Banach spaces

$$X = C^2([0, T], R^n) \times C^2([0, T], R^n)$$

and $$Y = L^1([0, T], R^n) \times L^1([0, T], R^n)$$

with the following norms : $\left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X = \sup_{t \in [0, T]} \sum_{i=1}^{n} (|u_i(t)| + |v_i(t)|)$ and $\left\| \begin{bmatrix} \tau \\ \eta \end{bmatrix} \right\|_Y = \int_0^T \sum_{i=1}^{n} (|\tau_i(t)| + |\eta_n(t)|)\, dt$, for $\begin{bmatrix} u \\ v \end{bmatrix} \in X \begin{bmatrix} \tau \\ \eta \end{bmatrix} \in Y.$

Clearly $X \subset Y$. We define the bilinear pairing between $X \subset Y$ to the reals as

$$(x, y) = \int_0^T [u(t) \circ \tau(t) + v(t) \circ \eta(t)]\, dt$$

in which '$\circ$' denotes the usual Euclidean inner product in R^n, for $\begin{bmatrix} u \\ v \end{bmatrix} \in X$, $\begin{bmatrix} \tau \\ \eta \end{bmatrix} \in Y$. This pairing satisfies the condition

$$|(x, y)| \leq \| x \|_X \| y \|_Y.$$

We also denote by $|\ |$, any appropriate norm in R^n. Let Y_2 be the subspace of Y defined by

$$Y_2 = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} Y / u(t) = \text{const.}, v(t) = \text{const. a. e on } [0, T] \right\}$$

and let Y_1 be the subspace of Y such that $Y \cong Y_1 \oplus Y_2$. Further, we define the Canonical projection operators :

$P : Y \to Y_1$ and $Q : Y \to Y_2$ by

$$P\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u(t) - \frac{1}{T}\int_0^T u(s)\,ds \\ v(t) - \frac{1}{T}\int_0^T v(s)\,ds \end{bmatrix} \qquad \ldots (3.3)$$

$$Q\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\int_0^T u(s)\,ds \\ \frac{1}{T}\int_0^T v(s)\,ds \end{bmatrix} \qquad \ldots (3.4)$$

for $\begin{bmatrix} u \\ v \end{bmatrix} \in Y$. Clearly $Q = I - P$, where I denotes the identity mapping on Y, and the projections P and Q are continuous. Let $X_2 = X \bigcap Y_2$. It is easy to see that X_2 is a closed subspace of X. Let X_1 be the closed subspace of X such that $X = X_1 \oplus X_2$. We note that $P(X) \subset X_1$, $Q(X) \subset X_2$ and the projections $P/X : X \to X_1$ $Q\,|_X : X \to X_2$ are continuous. Also we note that for $x = \begin{bmatrix} u \\ v \end{bmatrix} \in X$, $y = \begin{bmatrix} \tau \\ \eta \end{bmatrix} \in Y$ so that $x = Px + Qx$ and $y = Py + Qy$ we have

$$(x, y) = (Px, Py) + (Qx, Qy).$$

We shall now for the sake of convenience state the following conditions :

(C_1) There exist constants $\alpha > 0$, $\rho \geq 0$ and a non-negative function $\beta(t) \in L^1\,[0, T]$ such that

$$u \circ f(t, u, v) + v \circ g(t, u, v)$$

$$\leq \beta(t)\,(1 + |u| + |v|) - \alpha\,(|u| + |v|)\,[|f(t, u, v)| + |g(t, u, v)|] \qquad \ldots (3.5)$$

for all $t \in [0, T]$ and $u, v \in R^n$ with $|u| + |v| \geq \rho$.

(C_2) There exists a constant $c > 0$ and a nonnegative function $\gamma(t) \in L^1\,[0, T]$ such that

$$u \circ f(t, u, v) + v \circ g(t, u, v) \leq \gamma(t) - c(|u| + |v|) \qquad \ldots (3.6)$$

for all $u, v \in R^n$ and $t \in [0, T]$.

(C_3) For every $k > 0$, there exist nonnegative functions $f_k(t)$ and $g_k(t)$ in $L^1\,[0, T]$ such that

$$|f(t, u, v)| \leq f_k(t) \text{ for } u, v \in R^n \text{ with } |u| \leq k$$

and $$|g(t, u, v)| \leq g_k(t) \text{ for } u, v \in R^n \text{ with } |v| \leq k,\ t \in [0, T].$$

(C_4) There exist constant $\alpha > 0$, $\rho \geq 0$ and a nonnegative function $\beta(t) \in L^1[0, T]$ such that

$$u \circ f(t, u, v) + v \circ g(t, u, v) \leq \beta(t)(1 + Lu| + |v|) - \alpha(|u||f(t, u, v)| + |v|\,|g(t, u, v)|) \quad \ldots (3.7)$$

for all $t \in [0, T]$ and $u, v \in R^n$ with $\max(|u|, |v|) \geq \rho$.

We shall now prove the following :

Theorem 3.1 — Assume that (C_1) and (C_2) hold. Then the PBVB (3.1)-(3.2) has atleast one solution on $[0, T]$.

PROOF : We define the linear operator $L : D(L) \subset X \to Y$ by

$$L\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u'' \\ -v'' \end{bmatrix}. \quad \ldots (3.8)$$

Here $$D(L) = \left\{ \begin{bmatrix} \tau \\ \eta \end{bmatrix} \in X : \tau(0) = \tau(T), \tau'(0) = \tau'(T) \right.$$

$$\eta(0) = \eta(T), \eta'(0) = \eta'(T).$$

It may be seen that L is a linear Fredholm mapping with Ker $L = X_2$ and $R(L) = Y_1$. Also let $K : Y_1 \to X_1$ be given by

$$K\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\int_0^t (t-s)\,u(s)\,ds - \frac{t}{T}\int_0^T su(s)\,ds \\ + \frac{1}{T}\int_0^T \left[\int_0^t (t-s)u(s)ds + \frac{t}{T}\int_0^T su(s)ds \right] dt \\ -\int_0^t (t-s)v(s)\,ds - \frac{t}{T}\int_0^T sv(s)ds \\ + \frac{1}{T}\int_0^T \left[\int_0^T (t-s)v(s)ds + \frac{t}{T}\int_0^T sv(s)ds \right] dt \end{bmatrix}.$$

for $\begin{bmatrix} u \\ v \end{bmatrix} \in Y_1$.

It is easy to see that

$$LKP\begin{bmatrix} u \\ v \end{bmatrix} = P\begin{bmatrix} u \\ v \end{bmatrix} \text{ and } \left(KP\begin{bmatrix} u \\ v \end{bmatrix}, P\begin{bmatrix} u \\ v \end{bmatrix} \right) \geq 0, \text{ for } \begin{bmatrix} u \\ v \end{bmatrix} \in Y.$$

Further we define the nonlinear mapping N by

$$N\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -f(t, u, v) \\ -g(t, u, v) \end{bmatrix} \quad \ldots (3.9)$$

for $\begin{bmatrix} u \\ v \end{bmatrix} \in X$. Using Ascoli-Arzela theorem we see that the mapping $KPN : X \to X_1$ is continuous and compact.

Thus we see that L satisfies the conditions of Theorem 2.1 and that eqn. (3.1) can be written as

$$L\begin{bmatrix} u \\ v \end{bmatrix} + N\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} \qquad \text{... (3.10)}$$

in X. In order to conclude that (3.10) has at least one solution in X, it suffices to show that N satisfies conditions (H_1) and (H_2) of Theorem 2.1.

In view of (C_2), for $\begin{bmatrix} u \\ v \end{bmatrix} \in X$, we have

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) = \int_0^T [-u(t) \circ f(t, u(t), v(t)) - v(t) \circ g(t, u(t), v(t))]\, dt$$

$$\geq \int_0^T c\,(|u(t)| + |v(t)|)\, dt - \int_0^T \gamma(t)\, dt$$

$$\geq cT \left\| Q\begin{bmatrix} u \\ v \end{bmatrix} \right\|_X - d$$

where $d = \int_0^T \gamma(t)\, dt$ and from this we see that (H_1) of Theorem 2.1 is satisfied.

For $\begin{bmatrix} u \\ v \end{bmatrix} \in X$ and $k \geq \alpha\,\rho$, we have

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) = \int_0^T [-u(t) \circ f(t, u(t), v(t)) - v(t) \circ g(t, u(t), v(t))]\, dt$$

$$= -\int_{|u(t)| + |v(t)| \geq k/\alpha} [u(t) \circ f(t, u(t), v(t)) + v(t) \circ g(t, u(t), v(t))]\, dt$$

$$- \int_{|u(t)| + |v(t)| < k/\alpha} [u(t) \circ f(t, u(t), v(t)) + v(t) \circ g(t, u(t), v(t))]\, dt$$

$$= I_1 + I_2.$$

Clearly, we can find a constant $c(k)$ such that $|I_2| \leq c(k)$.

Further, (C_1) implies that

$$I_1 \geq \alpha \int_{|u(t)|+|v(t)| \geq k/\alpha} (|u(t)|+|v(t)|)\,[|f(t,u,(t),v(t))|+|g(t,u(t),v(t))|]\,dt$$

$$-\int_{|u(t)|+|v(t)| \geq k/\alpha} (1+|u(t)|+|v(t)|)\,\beta(t)\,dt$$

$$\geq k \left\| N\begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y - \beta' \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X - c'(k)$$

where $\beta' = \int_0^T \beta(t)dt$ and $c'(k)$ is a constant that depends on k.

Hence,

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) \geq k \left\| N\begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y - \beta' \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X - \bar{c}(k). \quad \ldots (3.11)$$

If $k < \alpha\rho$, then with $k = \alpha\rho$, we have

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) \geq \alpha\rho \left\| N\begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y - \beta' \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X - \bar{c}(\alpha)$$

$$\geq k \left\| N\begin{bmatrix} u \\ v \end{bmatrix} \right\|_Y - \beta' \left\| \begin{bmatrix} u \\ v \end{bmatrix} \right\|_X - \hat{c}(k).$$

Hence (3.11) holds for all $k > 0$ and thus N satisfies condition (H_2) of Theorem 2.1 and this completes the proof.

Theorem 3.2 — Assume that the conditions $(C_2), (C_3)$ and (C_4) hold. Then the PBVP (3.1)-(3.2) has atleast one solution on $[0, T]$.

PROOF : Following the proof of Theorem 3.1, we see that the condition (H_1) of Theorem 2.1 follows at once.

To verify the condition (H_2) of Theorem 2.1, we let $k \geq \alpha\rho$ and $\begin{bmatrix} u \\ v \end{bmatrix} \in X$. Then

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix} \right) = \int_0^T [-u(t)\circ f(t,u(t),v(t)) - v(t)\circ g(t,u(t),v(t))]\,dt$$

$$= -\int_{\max(|u(t)|,|v(t)|) \geq k/\alpha} [u(t)\circ f(t,u(t),v(t)) + v(t)\circ g(t,u(t),v(t))]\,dt$$

$$-\int_{\max(|u(t)|,|(t)|) < k/\alpha} [u(t)\circ f(t,u(t),v(t)) + v(t)\circ g(t,u(t),v(t))]\,dt$$

$$= I_1 + I_2 \qquad \text{(say)}.$$

We note that $|I_2| \leq a$ constant that depends on k. In view of (3.7), we have

$$I_1 \geq \alpha \int_{\max((|u(t)|, |v(t)|) \geq k/\alpha} [|u(t)||f(t, u(t), v(t))| + |v(t)||g(t, u(t), v(t))|]\, dt$$

$$- \int_{\max(|u(t)|, |v(t)|) \geq k/\alpha} \beta(t)(1 + |u(t)| + |v(t)|)\, dt$$

$$\geq \alpha \int_{|u(t)| \geq k|\alpha} |u(t)||f(t, u(t), v(t))|\, dt$$

$$+ \alpha \int_{\max|u(t)| < k/\alpha,\ |v(t)| \geq k/\alpha} |u(t)||f(t, u(t), v(t))|\, dt$$

$$+ \alpha \int_{|v(t)| \geq k/\alpha} |v(t)||g(t, u(t), v(t))|\, dt$$

$$+ \alpha \int_{|v(t)| < k/\alpha,\ |u(t)| \geq k/\alpha} |v(t)||g(t, u(t), v(t))|\, dt$$

$$- \int_0^T \beta(t)(1 + |u(t)| + |v(t)|)dt$$

$$\geq k \left|\left| N\begin{bmatrix} u \\ v \end{bmatrix} \right|\right|_Y - \beta' \left|\left| \begin{bmatrix} u \\ v \end{bmatrix} \right|\right|_X - \text{const.}\,(k).$$

Notice that in the above inequality, we have used the fact that there exist function $f_{k/\alpha}, g_{k/\alpha} \in L^1[0, T]$. Thus

$$\left(N\begin{bmatrix} u \\ v \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right) \geq k \left|\left| N\begin{bmatrix} u \\ v \end{bmatrix} \right|\right|_Y - \beta' \left|\left| \begin{bmatrix} u \\ v \end{bmatrix} \right|\right|_X - \text{const.}\,(k) \quad \text{... (3.12)}$$

for $k \geq \rho$. It is easy to see that the inequality (3.12) holds for every $k \geq 0$ as in the proof of Theorem 3.1. This completes the proof.

Remark 3.3 : Define an operator $L^* : D(L^*) \subset X \to Y$ by setting

$$D(L^*) = \left\{ \begin{bmatrix} u \\ v \end{bmatrix} \Big/ u(0) = u(T), u'(0) = u'(T), v(0) = v(T), v'(0) = v'(T) \right\}$$

and for $\begin{bmatrix} u \\ v \end{bmatrix} \in D(L^*)$,

$$L^*\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} v'' \\ -u'' \end{bmatrix}. \quad \text{... (3.13)}$$

Then L^* is linear Fredholm mapping with X_2 = Ker L^* and $Y_1 = R(L^*)$, (the spaces X, Y, X_1, X_2, Y_1, Y_2 are as defined earlier).

The mapping $K^* : Y_1 \to X_1$ defined by

$$K^* \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} \int_0^t (t-s)\, v(s)\, ds + \frac{t}{T} \int_0^T s v(s)\, ds \\ \qquad - \frac{1}{T} \int_0^T \left[\int_0^t (t-s) v(s) ds + \frac{t}{T} \int_0^T s v(s) ds \right] dt \\ - \int_0^t (t-s) u(s)\, ds - \frac{t}{T} \int_0^T s u(s) ds \\ \qquad + \frac{1}{T} \int_0^T \left[\int_0^t (t-s) u(s) ds + \frac{t}{T} \int_0^T s u(s) ds \right] dt \end{bmatrix}$$

is bounded and linear. Furthermore,

$$L^* K^* P \begin{bmatrix} u \\ v \end{bmatrix} = P \begin{bmatrix} u \\ v \end{bmatrix} \text{and} \left(K^* P \begin{bmatrix} u \\ v \end{bmatrix}, P \begin{bmatrix} u \\ v \end{bmatrix} \right) \geq 0 \text{ for } u, v \in R^n.$$

In view of this and the operator N^* defined by

$$N^* \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -g(t, u, v) \\ -f(t, u, v) \end{bmatrix},$$

we observe that the conclusion of Theorem 3.1 holds for the following system:

$$-u'' = f(t, u, v) + p, \; v'' = g(t, u, v) + q, \qquad \text{... (3.14)}$$

$$u(0) = u(T), u'(0) = u'(T), v(0) = v(T), v'(0) = v'(T),$$

when the inequalities (3.5) and (3.6) are replaced respectively by the conditions :

$$u \circ g(t, u, v) + v \circ f(t, u, v) \leq -\alpha(|u| + |v|)\; (|g(t, u, v)| + |f(t, u, v)|)$$
$$+ \beta(t)(1 + |u| + |v|) \qquad \text{... (3.15)}$$

for $t \in [0, T]$, $u, v \in R^n$ with $|u| + |v| \geq \rho$, and

$$u \circ g(t, u, v) + v \circ f(t, u, v) \leq -c\,(|u| + |v|) + \gamma(t) \qquad \text{... (3.16)}$$

for $t \in [0, T]$ and $u, v \in R^n$.

Similar remark is valid for the conclusion of Theorem 3.2.

Corollary 3.4 — Let $f(t, u, v) = f(t, v)$, $g(t, u, v) = g(t, u)$ be R^n-valued continuous functions on $[0, T] \times R^n$. Suppose that there exist constants $\alpha > 0, \rho \geq 0$ and a nonnegative function $\beta(t) \in L^1[0, T]$ such that

$$u \circ g(t, u) + v \circ f(t, v) \leq -\alpha\,(|u|\,|g(t, u)| + |v|\,|f(t, v)|)$$
$$+ \beta(t)(1 + |u| + |v|) \qquad \text{... (3.17)}$$

for $u, v \in R^n$ with $|u|+|v| \geq \rho$, $t \in [0, T]$. Further, suppose that there exists a constant $c > 0$ and a nonnegative function $\gamma(t) \in L^1[0, T]$ such that

$$u \circ g(t, u) + v \circ f(t, v) \leq -c\,(|u|+|v|) + \gamma(t) \qquad \text{... (3.18)}$$

for $u, v \in R^n$, $t \in [0, T]$.

Then for any pair of continuous functions $p(t$, $q(t)$ on $[0, T]$ satisfying $\int_0^T p(t)\,dt = \int_0^T q(t)\,dt = 0$, there exists at least one pair of functions $u(t), v(t) \in C^2([0, T], R^n)$ satisfying the system

$$\begin{aligned} -u'' &= f(t, v) + p(t), \\ v'' &= g(t, u) + q(t), \; t \in [0, T], \qquad \text{... (3.19)} \\ u(0) &= u(T), u'(0) = u'(T), v(0) = v(T), v'(0) = v'(T). \end{aligned}$$

The proof of this corollary follows at once from Theorem 3.2.

Remark 3.5 : If the functions $f(t, v)$ and $g(t, u)$ in corollary 3.4 are real valued functions on $[0, T] \times R$ such that $f(t, v)$ nondecreasing in v and $g(t, u)$ is nondecreasing in u, then (3.17) is satisfied.

Remark 3.6 : If the functions $f(t, u, v), g(t, u, v), p(t), q(t)$ in Theorem 3.1 or in Theorem 3.2 are periodic functions of period T, that is $f(0, u, v) = f(T, u, v)$ $g(0, u, v) = g(T, u, v)$, $p(0) = p(T)$, $q(0) = q(T)$, for u, $v \in R^n$, then it follows that the system (3.1) has atleast one pair $(u(t), v(t))$, $u(t), v(t) \in C^2[0, T]$ as a solution that satisfies $u(0) = u(T)$, $u'(0) = u'(T)$, $u''(0) = u''(T)$, $v(0) = v(T)$, $v'(0) = v'(T)$, $v''(0) = v''(T)$. That is, we have obtained T-periodic solutions for the system (3.1), though we have not studied the system (3.1) in the space of T-periodic functions on R.

We now present some results when the functions $f(t, u, v)$ and $g(t, u, v)$ are such that the corresponding operator $N : X \to Y, X, Y$ being Banach spaces as defined earlier, is monotone, that is for $u, v \in Y^*$, the dual of the space Y, we have

$$(Nu - Nv, u - v) \geq 0.$$

Theorem 3.7 — Let (C_2) and the following assumption hold.

(C_5) There exists a nonnegative function $\beta(t) \in L^1[0, T]$ such that for $t \in [0, T], u_i, v_i \in R^n$, $i = 1, 2, 3$, we have

$$\sum_{i=1}^{3} (u_i - u_{i+1}) \circ f(t, u_i, v_i) + \sum_{i=1}^{3} (v_i - v_{i+1}) \circ g(t, u_i, v_i) \leq \beta(t) \qquad \text{... (3.20)}$$

in which $u_4 = u_1$ and $v_4 = v_1$.

Then for any given pair of continuous functions $p(t)$, $q(t)$ on $[0, T]$ such that $\int_0^T p(t)\,dt = \int_0^T q(t)\,dt = 0$, the coupled PBVP (3.1)-(3.2) has atleast one solution in $C^2[0, T]$.

PROOF : Let X and Y be the Banach spaces as defined at the beginning of this Section. We note that X is a closed subspace of Y^*. The system (3.1) is equivalent to the operator equation

$$L \begin{bmatrix} u \\ v \end{bmatrix} + N \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

where L and N are as defined in the proof of Theorem 3.1. Since the functions $f(t, u, v)$, $g(t, u, v)$ are continuous on $[0, T] \times R^n \times R^n$, from (3.20), it follows that N is a bounded mapping from Y^* into Y and satisfies the condition (2.4) of Lemma 2.3 with $C(y, z) \equiv \int_0^T \beta(t)\, dt$ for all $y, z \in Y^*$.

Hence N satisfies condition (2.3) of Theorem 2.1. The other conditions on L and N of Theorem 2.1 follow from our assumptions as in the proof of Theorem 3.1. Thus the proof of the theorem is complete.

Corollary 3.8 — Let $f(t, u, v)$ be continuous and R^n-valued on $[0, T] \times R^n \times R^n$. Further, if

(i) there exists a nonnegative function $\beta(t) \in L^1[0, T]$ such that for $t \in [0, T], u_i\, v_i \in R^n$, $i = 1, 2, 3$,

$$\sum_{i=1}^{3} (u_i - u_{i+1}) \circ f(t, u_i, v_i) \leq \beta(t) \qquad \text{... (3.21)}$$

and

(ii) there exists a constant $c > 0$ and nonnegative function $\gamma(t) \in L^1[0, T]$ such that

$$u \circ f(t, u, v) \leq -c \,\|u\| + \gamma(t) \qquad \text{... (3.22)}$$

for $t \in [0, T]$, $u, v \in R^n$, then for any given continuous function $p(t)$ on $[0, T]$ with $\int_0^T p(t)\, dt = 0$, the following system

$$-u^{(4)} = f(t, u, u'') + p(t), \;\; t \in [0, T] \qquad \text{... (3.23)}$$

$$u(0) = u(T), u''(0) = u''(T)$$

has atleast one solution on $[0, T]$.

PROOF : It is easy to see that eqn. (3.23) is equivalent to the coupled system :

$$u'' = v,$$

$$-v'' = f(t, u, v) + p(t), \qquad \text{... (3.24)}$$

$$u(0) = u(T), v(0) = v(T).$$

Thus, the proof of this corollary follows from that of Theorem 3.7.

Remark 3.9 : We observe that the solution u of the equation (3.23) can be extended to a continuous periodic function of period T on R such that the derivative $u''(t)$ is also a continuous periodic function of period T on R. If we assume in addition that $f(t, u, v), p(t)$ are periodic functions of period T, then the fourth derivative $u^{(4)}(t)$ is also T-periodic.

ACKNOWLEDGEMENT

The author is very much thankful to Prof. V. Sree Hari Rao, Department of Mathematics, JNTU, Kukatpalli, Hyderabad for his helpful suggestions and encouragement.

REFERENCES

1. S. R. Bernfeld and V. Lakshmikantham, *An Introduction to Nonlinear Boundary Value Problems*, Academic Press, New York, 1974.
2. H. Brezis, *Proc. Int. Cong. Math., Vancouver* **2**(1974), 249-55.
3. H. Brezis and F. E. Browder, *Adv. Math.* **18**(1975), 115-47.
4. H. Brezis and F. E. Browder, *C.R. Acad. Sci., Paris*, t. **279**(1974), Ser. A., 1-2.
5. B. Calvert and C. P. Gupta, *Nonlinear Analysis Theory Methods and Applications* **2** (1976), 1-26.
6. R. Gaines and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer-Lecture Notes. **568** (1970), 260p.
7. C. P. Gupta, *Proc. Int. Conference, Nonlinear Systems and Applications*, ed. V. Lakshmikantham, Academic Press, New York, (1977), pp. 547-60.
8. C. P. Gupta, *Nonlinear Analysis Theory Methods and Applications* **3** (1979), 213-27.
9. R. Kannan and V. Lakshmikantham, *Applicable Analysis* **17**(1984), 103-13.
10. V. Lakshmikantham, V. Sree Hari Rao and A. S. Vatsala, *App. Math. Comput.* **15** (1984), 71-83.
11. S. Leela, *Nonlinear Analysis Theory Methods and Applications* **7**(1983), 349-55.
12. K. Narsimha Reddy, *Indian J. pure appl. Math.* **20**(1989), 1213-29.
13. V. Sree Hari Rao and A. S. Vatsala, *Applicable Analysis* **17**(1983), 23-36.