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GRAPHS WITH UNIQUE MINIMUM GRAPHOIDAL COVER

S. ARUMUGAM[1] AND C. PAKKIAM[2]

[1]*Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli* 627 009

[2]*V. O. C. College, Tuticorin 628 008*

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A graphoidal cover of a graph G is a collection ψ of (not necessarily open) paths in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most one path in ψ and every edge of G is in exactly one path in ψ. The graphoidal covering number γ of G is defined to be the minimum cardinality of a graphoidal cover of G. A graphoidal cover ψ of G with $|\psi| = \gamma$ is called a minimum graphoidal cover of G. Two minimum graphoidal covers ψ_1 and ψ_2 are said to be isomorphic if there exists an automorphism α of G such that $\psi_2 = \{\alpha(P)/P \in \psi_1\}$. In this paper we study the properties of graphs in which any two minimum graphoidal covers are isomorphic.

1. INTRODUCTION

By a graph we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary[2]. All graphs considered in this paper are assumed to be connected graphs without isolates. The order and size of a graph G are denoted by p and q respectively. The concept of graphoidal cover was introduced by Acharya and Sampathkumar[1].

Definition 1.1 — A graphoidal cover of a graph $G = (V, E)$ is a set ψ of (not necessarily open) paths in G satisfying the following conditions.

(i) Every path in ψ has at least two vertices.

(ii) Every vertex of G is an internal vertex of at most one path in ψ.

(iii) Every edge of G is in exactly one path in ψ.

The graphoidal covering number γ of G is defined to be the minimum cardinality taken over all graphoidal covers of G. Any graphoidal cover ψ of G with $|\psi| = \gamma$ is called a minimum graphoidal cover of G. In this paper we introduce the concept of isomorphism between graphoidal covers and study the properties of graphs in which any two minimum graphoidal covers are isomorphic. We need the following theorems in the sequel.

Theorem 1.2[3] — For any tree T with n pendent vertices $\gamma(T) = n - 1$.

Thoerem 1.3[3] — If there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of a path in ψ, then ψ is a minimum graphoidal cover of G and $\gamma(G) = q - p$.

Theorem 1.4[3] — If there exists a graphoidal cover ψ of G such that every vertex v with $d(v) > 1$ is an internal vertex of a path in ψ, then ψ is a minimum graphoidal cover of G and $\gamma(G) = q - p + n$ where n is the number of pendent vertices of G.

Theorem 1.5[4] — Let G be a unicyclic graph with n pendent vertices. Let C be the unique cycle in G and let m be the number of vertices of degree greater than 2 on C. Then

$$\gamma(G) = \begin{cases} 1 & \text{if } m = 0 \\ n + 1 & \text{if } m = 1 \text{ and } d(v) = 3 \text{ where } v \text{ is the unique Cabove} \\ & \text{vertex of degree greater tahn 2 on} \\ n & \text{otherwise} \end{cases}$$

2. Main Results

Definition 2.1 — Two graphoidal covers ψ_1 and ψ_2 of a graph G are said to be isomorphic if there exists an automorphism α of G such that $\psi_2 = \{\alpha(P)/P \in \psi_1\}$. G is said to have a unique minimum graphoidal cover if any two minimum graphoidal covers of G are isomorphic.

Remark 2.2 : Let $\psi = \{P_1, P_2, ..., P_n\}$ be a graphoidal cover of G. Let k_i be the length of the path P_i and let $k_1 \leq k_2 \leq \ldots \leq k_n$. Since every edge of G is in exactly one P_i, we have $\sum_{i=1}^{n} k_i = q$. Thus every graphoidal cover of G gives rise to a partition of the integer q. Clearly any two isomorphic graphoidal covers of G give rise to the same partition of q. The following example shows that the converse is not true.

Example 2.3 — Consider the graph G given in Fig. 1.

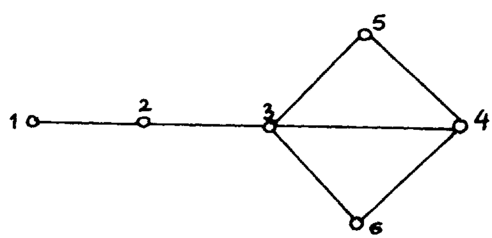

FIG. 1.

Let $\psi_1 = \{(1, 2, 3, 6, 4), (3, 4, 5, 3)\}$

and $\psi_2 = \{(3, 6, 4, 5, 3), (1, 2, 3, 4)\}$.

Since ψ_1 contains a path of length 4 and a cycle of length 3 and ψ_2 contains a path of length 3 and a cycle of length 4, ψ_1 and ψ_2 are not isomorphic. However ψ_1 and ψ_2 are minimum graphoidal covers of G which give rise to the same partition of q.

We now proceed to investigate properties of graphs in which any two minimum graphoidal covers are isomorphic.

Theorem 2.4 — Let G be a graph with unique minimum graphoidal cover. Then $\delta \leq 3$.

PROOF : Suppose $\delta \geq 4$. Let $P_1 = (u_1, u_2, ..., u_n)$ be a longest path in G so that all vertices adjacent to u_1 or u_n are already in P_1. Since $\delta \geq 4$, we can find vertices u_i, u_j such that $2 < i < j < n$ and u_i, u_j are adjacent to u_1. Similarly we can find vertices u_r, u_s in P_1 such that $1 < r < s < n - 1$ and u_r, u_s are adjacent to u_n. Now, let $P_2 = (u_i, u_1, u_j)$ and $P_3 = (u_r, u_n, u_s)$. Then $\{P_1, P_2, P_3\}$ is a collection of edgedisjoint and internally disjoint paths in which each of the vertices $u_1, u_2, ..., u_n$ is an internal vertex of one path in the collection. Now if $\{u_1, u_2, ..., u_n\} \neq V(G)$, let w_1 be a vertex not on P_1. Let P_4 be a longest path in G containing w_1 and internally disjoint with the paths P_1, P_2 and P_3. If the end points of P_4 are not in P_1, we make them internal vertices of some path as before. Continuing this process we obtain a collection of paths $\{P_1, P_2, ..., P_m\}$ in G which are edgedisjoint and every vertex of G is an internal vertex of exactly one P_i. Let S denote the set of all edges of G which are not covered by the paths P_i, $1 \leq i \leq m$. By Theorem 1.3, $\psi = \{P_1, P_2, ..., P_m\} \bigcup S$ is a minimum graphoidal cover of G. Now, let

$$Q_1 = (u_i, u_{i-1}, ..., u_1, u_j) \text{ and } Q_2 = (u_1, u_i, u_{i+1}, ..., u_n).$$

Then $\psi_1 = (\psi - \{P_1, P_2\}) \cup \{Q_1, Q_2\}$ is also a minimum graphoidal cover of G. Now the paths P_1 and P_2 have lengths $n - 1$ and 2 respectively whereas the paths Q_1 and Q_2 have lengths i and $n - i + 1$ respectively. Also $2 < i < n - 1$. Hence ψ and ψ_1 give rise to different partitions of q so that the two minimum graphoidal covers ψ and ψ_1 are nonisomorphic, which is a contradiction. Thus $\delta \leq 3$. ■

Theorem 2.5 — Let G be a graph with $\delta = 3$. Then G has a unique minimum graphoidal cover if and only if $G = K_4$.

PROOF : Suppose $G = K_4$ and let $V(G) = \{1, 2, 3, 4\}$. Then $\psi = \{(1, 2, 3, 4), (2, 4, 1, 3)\}$ is a minimum graphoidal cover for G. Further any minimum graphoidal cover of G consists of two edgedisjoint paths of length 3. Hence any two minimum graphoidal covers of G are isomorphic.

Conversely, let G be a graph with $\delta = 3$ in which any two minimum graphoidal covers are isomorphic. Let $P_1 = (u_1, u_2, \dots, u_n)$ be a longest path in G so that all vertices adjacent to u_1 or u_n are already in P_1. Since $\delta = 3$, we have $n \geq 4$. We claim that $n = 4$. Suppose $n \geq 5$.

Case i — u_1 and u_n are nonadjacent.

Since $\delta = 3$ we can find vertices u_i, u_j and u_r, u_s in P_1 such that $2 < i < j < n$, u_i and u_j are adjacent to u_1 and $1 < r < s < n - 1$, u_r and u_s are adjacent to u_n.

Now, let $P_2 = (u_i, u_1, u_j)$ and $P_3 = (u_r, u_n, u_s)$. Proceeding as in Theorem 2.4, we obtain two nonisomorphic minimum graphoidal covers of G, which is a contradiction.

Case ii — u_1 and u_n are adjacent.

Since $\delta = 3$ we can find vertices u_i, u_j on P_1 such that $2 < i < n$, $1 < j < n - 1$, u_i is adjacent to u_1 and u_j is adjacent to u_n. Without loss of generality we assume that $i \leq j$. Now, let $P_2 = (u_i, u_1, u_n, u_j)$. Then $\{P_1, P_2\}$ is a collection of edgedisjoint paths in which each of the vertices $u_1, u_2, \ldots, u_n$ is an internal vertex of exactly one of these paths. Proceeding as in Theorem 2.4 we construct a collection of paths $\{P_1, P_2, ..., P_m\}$ in G which are edgedisjoint and every vertex of G is an internal vertex of exactly one P_i. Let S denote the set of all edges of G not covered by the paths P_k, $1 \leq k \leq m$. By Theorem 1.3, $\psi = \{P_1, P_2, ..., P_m\} \cup S$ is a minimum graphoidal cover of G.

Now if $i < j$, let

$$Q_1 = (u_i, u_{i-1}, ..., u_1, u_n, u_j)$$

and $$Q_2 = (u_n, u_{n-1}, ..., u_j, u_{j-1}, ..., u_i, u_1).$$

If $i = j$, let $Q_1 = (u_i, u_{i+1}, ..., u_n, u_1, u_2, \ldots, u_i)$ and $Q_2 = (u_1, u_i, u_n)$.

Then $\psi_1 = (\psi - \{P_1, P_2\}) \cup \{Q_1, Q_2\}$ is also a minimum graphoidal cover of G and ψ, ψ_1 give rise to two different partitions of q. Hence ψ and ψ_1 are nonisomorphic which is a contradiction.

Hence $n = 4$ and $P_1 = (u_1, u_2, u_3, u_4)$ is a longest path in G. Since $\delta = 3$, u_3, u_4 are adjacent with u_1 and u_1, u_2 are adjacent with u_4. Hence the subgraph of G induced by the vertices u_1, u_2, u_3, u_4 is K_4. If $G \neq K_4$, then there is a vertex v in G adjacent to some u_i and hence G contains a path of length greater than 3, which is a contradiction. Thus $G = K_4$. ■

Theorem 2.6 — A tree T has a unique minimum graphoidal cover if and only if there exists at most one vertex v with $d(v) > 2$ and the distance from v to all the pendent vertices of T are equal.

PROOF : If there is no vertex v with $d(v) > 2$, then T is a path and the result is trivial. Suppose G has a unique vertex v with $d(v) = k > 2$ and $d(v, v_i) = m$ for all pendent vertices v_i. Then any minimum graphoidal cover of T consists of one path of length $2m$ and $k - 2$ paths of length m and hence any two minimum graphoidal covers of T are isomorphic.

Conversely, let T be any tree in which any two minimum graphoidal covers are isomorphic. Let k denote the number of vertices of degree greater than 2. Suppose $k > 1$. Let $v_1, v_2, ..., v_k$ be the vertices of T with $d(v_i) > 2$. Let $(P_1, P_2, ..., P_k)$ be a collection of paths in T such that each P_j is a longest path having v_j as an internal vertex and internally disjoint and edgedisjoint with $P_1, P_2, ..., P_{j-1}$. Clearly the end vertices of each of these paths P_i are pendent vertices of T or the vertices $v_1, v_2, ..., v_k$. If there exists a vertex v with $d(v) = 2$ such that v is not an internal vertex of any of the paths $P_1, P_2, ..., P_k$, we choose a longest path P_{k+1} containing

v as an internal vertex and internally disjoint with $P_1, P_2, ..., P_k$. Proceeding like this we get a collection of paths $\{P_1, P_2, ..., P_m\}$ such that every vertex with degree greater than 1 is an internal vertex of exactly one of these paths. Let S denote the set of all edges not covered by the above paths. By Theorem 1.4, $\psi = \{P_1, P_2, ..., P_n\} \cup S$ is a minimum graphoidal cover of T such that every path in ψ has at most one v_i as an internal vertex. Now, let Q_1 be a longest path in T having both v_1 and v_2 as internal vertices. Let ψ_1 be a minimum graphoidal cover of T such that $Q \in \psi_1$. Clearly the minimum graphoidal covers ψ and ψ_1 of T are nonisomorphic which is a contradiction. Hence $k = 1$. Let v be the unqiue vertex of T with $d(v) = m > 2$. Then T has exactly m pendent vertices, say, $w_1, w_2, ..., w_m$. Suppose $d(v, w_1) \neq d(v, w_2)$. Let P_1 be the unique $w_3 - w_1$ path in T and let P_2 be the unique $w_3 - w_2$ path in T. Let Q_i be the unique $w_i - v$ path in $T (1 \leq i \leq m)$. Let $\psi = \{P_1, Q_2, Q_4, ..., Q_m\}$ and $\psi_1 = \{P_2, Q_1, Q_4, ..., Q_m\}$. Clearly ψ and ψ_1 are minimum graphoidal covers of T which give rise to two different partitions of q and hence are nonisomorphic, which is a contradiction. Thus the distance from v to all the pendent vertices of T are equal. ■

If $G = C_n$, a cycle on n vertices, then $\gamma(G) = 1$ and any two minimum graphoidal covers of G are isomorphic. The following theorem gives a characterisation of all unicyclic graphs, other than cycles, having a unique minimum graphoidal cover.

Theorem 2.7 — Let G be unicyclic graph with unique cycle $C = (v_1, v_2, ..., v_k, v_1)$ and let $G \neq C$. Let m denote the number of vertices v_i on C such that $d(v_i) \geq 3$. Then G has a unique minimum graphoidal cover if and only if $m = 1$, the unique vertex say v_1 on C with $d(v_1) \geq 3$ is such that $d(v_1) \geq 4$, all vertices not on C have degree 1 or 2 and the distances from v_1 to all the pendent vertices of G are equal.

PROOF : Let n denote the number of pendent vertices of G. Suppose that G satisfies the conditions of the theorem. Let s denote the distance from v_1 to any pendent vertex in G. Any minimum graphoidal cover ψ of G consists of the cycle $C = (v_1, v_2, ..., v_n, v_1)$, a path of length $2s$ joining two pendent vertices of G which contains v_1 as an internal vertex and $n - 2$ paths of length s joining v_1 to each of the remaining $n - 2$ pendent vertices. Clearly any two such minimum graphoidal covers of G are isomorphic.

Conversely, let G be a unicyclic graph with a unique minimum graphoidal cover. Suppose $m > 1$. Let v_1, v_i $(2 \leq i \leq k)$ be two vertices on C such that $d(v_1), d(v_i) \geq 3$. Let $P = (v_1, w_1, w_2, ..., w_r)$ be a longest path in G such that v_1 is the only vertex common to P and C. Let $Q = (v_i, z_1, z_2, ..., z_s)$ be a longest path in G such that v_i is the only vertex common to Q and C. Since G is unicyclic, P and Q are vertex disjoint paths.

Let $\quad P_1 = (w_r, w_{r-1}, ..., w_1, v_1, v_2, ..., v_i, z_1, z_2, ..., z_s),$

$P_2 = (v_1, v_n, v_{n-1}, ..., v_i),$

$Q_1 = (w_r, w_{r-1}, ..., w_1, v_1, v_2, ..., v_i),$

and $\quad Q_2 = (v_1, v_n, ..., v_i, z_1, z_2, ..., z_s).$

Clearly the two collection of paths $\{P_1, P_2\}$ and $\{Q_1, Q_2\}$ cover the same set of edges, have the same set of vertices as internal vertices, these paths cannot be extended further to cover more edges and these edges cannot be covered by a fewer number of paths. Hence we can find paths $P_3, P_4, ..., P_n$ in G such that $\psi_1 = \{P_1, P_2, P_3, ..., P_n\}$ and $\psi_2 = \{Q_1, Q_2, P_3, ..., P_n\}$ are minimum graphoidal covers of G. Clearly ψ_1 and ψ_2 determine two different partitions of q and hence are nonisomorphic which is a contradiction. Thus $m = 1$.

Let v_1 be the unique vertex on C such that $d(v_1) \geq 3$. Suppose $d(v_1) = 3$. Then $G_1 = G - \{v_2, v_3, ..., v_k\}$ and $G_2 = G - \{v_1, v_2\}$ are trees with $n + 1$ pendent vertices. Hence by Theorem 1.2, $\gamma(G_1) = \gamma(G_2) = n$. Let ψ_1 and ψ_2 be minimum graphoidal covers of G_1 and G_2 respectively. Then $\psi_3 = \psi_1 \cup \{(v_1, v_2, ..., v_n, v_1)\}$ and $\psi_4 = \psi_2 \cup \{\{v_1, v_2\}\}$ are graphoidal covers of G and $|\psi_3| = |\psi_4| = n + 1$. By Theorem 1.5, ψ_3 and ψ_4 are minimum graphoidal covers of G. Since one member of ψ_3 is a cycle and every member of ψ_4 is a path, ψ_3 and ψ_4 are not isomorphic which is a contradiction. Hence $d(v_1) \geq 4$ and by Theorem 1.5, $\gamma(G) = n$.

Now $G_1 = G - \{v_2, v_3, ..., v_k\}$ is a tree with $n - 1$ pendent vertices and any minimum graphoidal cover ψ of G is of the form $\psi = \psi_1 \cup \{C\}$ where ψ_1 is a minimum graphoidal cover of G_1. Since G has a unique minimum graphoidal cover, G_1 also has a unique minimum graphoidal cover. Hence by Theorem 2.6 there exists at most one vertex w in G_1 with $d_{G_1}(w) \geq 3$ and the distance from w to all pendent vertices of G_1 are equal. We distinguish two cases :

Case i — There exists a vertex w in G_1 such that $d_{G_1}(w) \geq 3$. Let $d_{G_1}(w, v) = a$, where v is any pendent vertex of G_1.

We claim that $w = v_1$. Supoose $w \neq v_1$. Let ψ_1 be a minimum graphoidal cover of G_1 in which v_1 lies on a path of length $2a$ and let ψ_2 be a minimum graphoidal cover of G_1 in which v_1 lies on a path of length a. Then $\psi_1 \cup \{C\}$ and $\psi_2 \cup \{C\}$ are nonisomorphic minimum graphoidal covers of G, which is a contradiction. Thus $w = v_1$ so that every vertex not on C has degree 1 or 2 and the distance from v_1 to all the pendent vertices of G are equal.

Case ii — $d_{G_1}(v) \leq 2$ for all $v \in V(G_1)$.

In this case G_1 is a path $P = (w_1, w_2, ..., w_r, v_1, w_{r+1}, ..., w_s)$ having v_1 as an internal vertex and exactly two pendant vertices w_r and w_s. Hence all vertices not on C have degree 1 or 2. If $d(v, w_1) \neq d(v, w_2)$, then any automorphism α of G fixes every vertex of P and hence $\{C, P\}$ and $\{C, Q\}$ where $Q = (w_s, ..., v_{r+1}, v_1, w_r, w_{r-1}, ..., w_1)$ are two nonisomorphic minimum graphoidal covers of G, which is a contradiction. Hence $d(v, w_r) = d(v, w_s)$. ■

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