Indian J. pure appl. Math., 25(11): 1163-1170, November 1994

ON THE TOPOLOGY GENERATED BY SEMI-REGULAR SETS

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(*Received* 17 *August* 1993; *after revision* 4 *March* 1994; *accepted* 21 *April* 1994)

In this paper we study compactness properties of spaces whose topologies are generated by the family of semi-open sets or the family of semi-regular sets of a given topological space (X, τ). In addition, we obtain characterizations of extremally disconnected spaces and show that the concepts of semi-compactness and semi-countable compactness coincide. We also prove that the family of semi-regular sets of a space constitutes a topology iff the corresponding semi-regularization space is locally indiscrete.

1. INTRODUCTION AND PRELIMINARIES

Semi open sets were introduced by Levine[10] and also independently by Njastad[13] as β-sets. A subset A in a topological space (X, τ) is called semi open (resp. semi closed) if $A \subseteq \text{cl int } A$ (resp. $\text{int cl } A \subseteq A$) or equivalently $A = G_A \cup N_A$ by the aid of an appropriate open set G_A and a nowhere dense set $N_A \subseteq \partial G_A$ (resp. $A = K_A - N_A$ where K_A is closed and $N_A \subseteq \partial K_A$). The family of all semi open (resp. semi closed) sets in the topological space (X, τ) is written by $SO(X, \tau)$ (resp. $SC(X, \tau)$). Arbitrary unions (resp. intersections) of semi open (resp. semi closed) sets are semi open (resp. semi closed). Therefore the semi interior (resp. semi closure) of a subset A, i.e. $\text{sint } A = \bigcup \{B \subseteq A : B \in SO(X, \tau)\}$ (resp. $\text{scl } A = \bigcap \{B \supseteq A : B \in SC(X, \tau)\}$ is semi open (resp. semi closed) for any subset A. Furthermore $A \in SO(X, \tau)$ (resp. $A \in SC(X, \tau)$) iff $A = \text{sint } A$ (resp. $A = \text{scl } A$). Andrijevic[1] has proved that $\text{sint } A = A \cap \text{cl int } A$ and $\text{scl } A = A \cup \text{int cl } A$. Thus if A is semi open then scl A is semi open since it is the union of two semi open sets and consequently

$$\text{scl sint } A = \text{sint (scl sint } A), \text{ sint scl } A = \text{scl (sint scl } A)$$

* The second author is partially supported by the Turkish Scientific and Technical Research Council TBAG-C2.

hold for any subset A. It is known that the family $SO(X, \tau)$ is a topology iff (X, τ) is an extremally disconnected space i.e. the closures of all open sets in (X, τ) are open or equivalently disjoint open sets in (X, τ) have disjoint closures[13]. Njastad[13] additionally proved that the family $\tau_\alpha = \{A \subseteq X : A \subseteq \text{int cl int} A\}$ is always a topology on X and satisfies $\tau \subseteq \tau_\alpha = \tau_{\alpha\alpha} \subseteq SO(X, \tau) = SO(X, \tau_\alpha)$. Furthermore $SO(X, \tau) = \tau_\alpha$ iff (X, τ) is extremally disconnected iff (X, τ_α) is extremally disconnected. Members of τ_α are called as α-sets or feebly open sets of (X, τ).

Following M. H. Stone we say that a subset A in a topological space (X, τ) is regular open (resp. regular closed) iff A = int cl A (resp. A = cl int A). The family of all regular open (resp. regular closed) sets of (X, τ) is written by $RO(X, \tau)$ (resp. $RC(X, \tau)$). This family is closed under the finite intersections (resp. finite unions). The topology generated by the family $RO(X, \tau)$ is written by τ_s and known as the semi-regularization of τ. The space (X, τ) is called semi-regular if $\tau = \tau_s$ holds. it is well known that int cl $A \in RO(X, \tau)$ and cl int $A \in RC(X, \tau)$ hold for any $A \subseteq X$. The relationships $\tau_{ss} = \tau_s$ and $RO(X, \tau) = RO(X, \tau_s)$ are known. The family of all clopen sets of (X, τ) is denoted by $CO(X, \tau)$, and the following relationships are known or are easy to observe :

$$CO(X, \tau_s) = CO(X, \tau) = RO(X, \tau) \cap RC(X, \tau).$$

For an arbitrary subset A of the space (X, τ), Maio and Noiri[11] have proved that the following statements are equivalent :

(1) There exists an $U \in RO(X, \tau)$ with $U \subseteq A \subseteq \text{cl}\, U$.

(2) A = sint scl A.

(3) $A \in SO(X, \tau) \cap SC(X, \tau)$.

The family $SO(X, \tau) \cap SC(X, \tau)$ is written by $SR(X, \tau)$ and its members are called the semi-regular sets of (X, τ). This family is generally not necessarily a topology on X and contains the family $RO(X, \tau) \cup RC(X, \tau)$. It is easy to see that a subset is semiopen iff its complement is semi closed and therefore we evidently have the following :

Observation : $A \in SR(X, \tau)$ iff $X - A \in SR(X, \tau)$.

The topology generated by the family $SR(X, \tau)$ (resp. $SO(X, \tau)$) is written by τSR (resp. τSO). Dorsett[2] has proved that the containments

$$\tau_s \subseteq \tau SR = (\tau SR)_s \subseteq (\tau SO)_s \subseteq \tau SO \subseteq \tau SOSR \subseteq \tau SOSO$$

hold for any space (X, τ). The primary aim of this note is to give some covering characterizations of the spaces $(X, \tau SO)$ and $(X, \tau SR)$. The conventional abbreviations iff for if and only if and e.d. for extremally disconnected are used throughout the paper. No separation axiom is assumed unless explicitly stated. In particularly regular spaces are not necessarily Hausdorff.

Definition 1 — A topological space (X, τ) is called

(1) S-closed (resp. countably S-closed[4]) if every cover (resp. countable cover) of X by the regular closed sets has a finite subcover.

(2) Almost e.d. if $\partial U = \text{cl}\, U - U$ is finite for every $U \in RO(X, \tau)$.

(3) Semi-compact[3] (resp. semi-countably compact) if every cover (resp. countable cover) of X by semi open sets has a finite subcover.

(4) R_0 if $x \in G \in \tau$ give $\text{cl}\{x\} \subseteq G$.

(5) Weakly Hausdorff[14] if (X, τ_s) is T_1 or equivalently

$$\{x\} = \bigcap \{K : x \in K \in RC(X, \tau)\}$$

for each $x \in X$.

Definition 2 — A point $x \in X$ is an e.d. point[5] in the space (X, τ) if there exists no $U \in RO(X, \tau)$ with $x \in \partial U$. A subset A is called semi-preopen[1] in (X, τ) if $A \subseteq \text{cl int cl}\, A$ holds and the family of all such subsets in (X, τ) is written by $SPO(X, \tau)$. Following Hodel[7] we say that a family $\mathcal{G}$ of non empty open subsets is a cellular family if its members are pairwise disjoint.

2. THE RESULTS

Proposition 1 — $A \in SR(X, \tau)$ iff there exists a $B \in SPO(X, \tau)$ with $A = \text{scl}\, B$.

PROOF : If A is semi-regular, then $A = \text{scl}\, A$ and $A \in SO(X, \tau) \subseteq SPO(X, \tau)$. Conversely let $A = \text{scl}\, B$ where $B \in SPO(X, \tau)$. Then we have $U = \text{int cl}\, B \subseteq \text{scl}\, B = A \subseteq \text{cl}\, B = \text{cl}\, U$. Hence $A \in SR(X, \tau)$ since $U \in RO(X, \tau)$.

Corollary (Maio and Noiri[11]) — If $A \in SO(X, \tau)$ then $\text{scl}\, A \in SR(X, \tau)$.

Proposition 2 — (i) (X, τ) is e.d. iff $SR(X, \tau) = CO(X, \tau)$.

(ii) $\{x\} \in SR(X, \tau)$ iff $\{x\} \in RO(X, \tau)$.

PROOF : Left to the reader.

Corollary 2 — The following are equivalent for any (X, τ) :

(1) (X, τ) is e.d.

(2) $\tau SR = \tau_s$.

(3) $\tau SR \subseteq \tau$.

PROOF : (1) $\Rightarrow$ (2) If (X, τ) is e.d., then $SR(X, \tau) = RO(X, \tau) \subseteq \tau_s$ and thus $\tau SR = \tau_s$ follows easily.

(3) $\Rightarrow$ (1) If $K \in RC(X, \tau)$, then $K \in SR(X, \tau) \subseteq \tau SR \subseteq \tau$, i.e. (X, τ) is e.d. The implication (2) $\Rightarrow$ (3) is only obvious.

Corollary 3 — Let (X, τ) be a semi-regular R_0 space. Then the following are equivalent.

(1) $(X, \tau SR)$ is compact.

(2) (X, τ) is S-closed and e.d.

(3) (X, τ) is compact and e.d.

PROOF : (1) $\Rightarrow$ (2) If $(X, \tau SR)$ is compact, then (X, τ) is evidently S-closed. Thus the required implication follows from the following result of Jankovic and Konstadilaki[8] : An S-closed space (X, τ) is e.d. iff (X, τ_s) is R_0.

(2) $\Rightarrow$ (3) One has only to remember that extremally disconnected semi-regular spaces are regular and regular S-closed spaces are compact.

(3) $\Rightarrow$ (1) Easily follows from the above corollary.

Proposition 3 — The following are equivalent for any (X, τ) :

(1) (X, τ) is e.d.

(2) $SR(X, \tau)$ is closed under finite unions.

(3) $SR(X, \tau)$ is closed under finite intersections.

PROOF : (3) $\Rightarrow$ (1) Take any $U \in RO(X, \tau)$ and a point $x \in \text{cl}\, U$. If $x \notin U$, then $U_1 = U \cup \{x\}$ and $U_2 = (X - \text{cl}\, U) \cup \{x\}$ are both of semi regular sets and thus $\{x\} = U_1 \cap U_2 \in SR(X, \tau)$ by (3). Proposition 2 then gives the contradiction $x \in \text{int cl}\, U = U$. Thus cl $U = U$ holds for each $U \in RO(X, \tau)$ i.e. extremally disconnectedness of (X, τ) follows. Other implications are obvious.

Remark : Recall that a space (X, τ) is said to be locally indiscrete if every open subset is closed i.e. $\tau \subseteq CO(X, \tau)$. Nieminen[12] has proved that the following conditions are equivalent : (1) (X, τ) is locally indiscrete, (2) (X, τ) is the topological sum of indiscrete spaces, (3) (X, τ) has a base which is a partition of X, (4) (X, τ) is an R_0 space and the intersection of any family of open sets is open. Now we have

Proposition 4 — The following are equivalent for any space (X, τ) :

(1) $SR(X, \tau)$ is a topology.

(2) $SR(X, \tau)$ is closed under arbitrary unions.

(3) $SR(X, \tau)$ is closed under arbitrary intersections.

(4) (X, τ_s) is locally indiscrete.

(5) $\tau_s = CO(X, \tau)$.

(6) $cl_{\tau_s}\{x\} \in CO(X, \tau)$ for each $x \in X$.

PROOF : (1) $\Rightarrow$ (2) $\Rightarrow$ (3), (4) $\Leftrightarrow$ (5) and (4) $\Leftrightarrow$ (6) are all obvious.

(3) $\Rightarrow$ (4) If $U \in \tau_s$, then $X - U = \text{cl}_{\tau_s}(X - U) = \bigcap \{F \in RC(X, \tau)) : X - U \subseteq F\}$ and thus $X - U \in SR(X, \tau)$ by (3). Since (X, τ) is e.d. by Proposition 3, we have $X - U \in RO(X, \tau)$ and so $\tau_s \subseteq CO(X, \tau_s) = CO(X, \tau)$. Hence $\tau_s = CO(X, \tau)$ is obtained.

(6) $\Rightarrow$ (1) It is clear that (X, τ_s) and consequently (X, τ) has to be e.d. since a topological space is e.d. if the closures of all singletons are open in this space. Thus we have $SR(X, \tau) = RO(X, \tau) = CO(X, \tau)$. We have to show that $SR(X, \tau) = RO(X, \tau)$ is closed under arbitrary unions after Proposition 3. Let $A = \bigcup \{G_\mu : \mu \in I, G_\mu \in RO(X, \tau)\}$ and let $x \in \text{cl}\, A = \text{cl}_{\tau_s} A$. If $x \notin A$, then $\text{cl}_{\tau_s}\{x\} \cap G_\mu = \phi$ for each $\mu \in I$ and we would have $x \notin \text{cl}_{\tau_s} A$ by (6). Thus $A \in CO(X, \tau) = RO(X, \tau)$.

Corollary 4 — $SR(X, \tau)$ is a topology iff $CO(X, \tau)$ is a topology and (X, τ) is e.d.

Proposition 5 — For any $x \in X$ in the space (X, τ), $\{x\} \in \tau SR$ iff $\{x\} \in RO(X, \tau)$ or x is not an e.d. point.

PROOF : Suppose that x is an e.d. point of X and satisfies $\{x\} \in \tau SR$. Then by the aid of appropriate semi-regular sets S_k and regular open sets U_k satisfying $U_k \subseteq S_k \subseteq \text{cl}\, U_k$ ($k = 1, 2, ..., n$) we have

$$\bigcap_{k=1}^{n} U_k \subseteq \{x\} = \bigcap_{k=1}^{n} S_k \subseteq \bigcap_{k=1}^{n} \text{cl}\, U_k$$

Then $\{x\} = \bigcap \{U_k : 1 \le k \le n\} \in RO(X, \tau)$ is obtained easily and the necessity follows. If conversely x is not an e.d. point, then there exists an $U \in RO(X, \tau)$ with $x \in \text{cl}\, U - U$. Then we have $\{x\} = (U \cup \{x\}) \cap ((X - \text{cl}\, U) \cup \{x\}) \in \tau SR$ since the two sets participating the above intersection are semi-regular. If $\{x\} \in RO(X, \tau)$, then $\{x\} \in \tau SR$ follows obviously.

Corollary 5 — $(X, \tau SR)$ is discrete if (X, τ) has no e.d. point.

Lemma — Let $\mathcal{G} = \{G_\mu\}_{\mu \in I}$ be any infinite cellular family in a space in which all nowhere dense subsets are finite. Then there exists an index $\mu_0 \in I$ such that

$$\text{cl} \left(\bigcup_{\mu \in I} G_\mu \right) = G_{\mu_0} \cup \text{cl} \left(\bigcup_{\mu \neq \mu_0} G_\mu \right).$$

PROOF : Write $G = \bigcup \mathcal{G}$. Suppose that there exists

$$x_\mu \in \text{cl}\, G - (G_\mu \cup \text{cl} \left(\bigcup_{\beta \neq \mu} G_\beta \right)$$

for each $\mu \in I$. Then it is not difficult to see that $x_\mu \in \partial G \cap \partial G_\mu$ and thus $x_\mu \neq x_\beta$ holds whenever $\mu \neq \beta$. Thus the finite set ∂G would contain infinitely many different points, a contradiction.

Proposition 6 — The following are equivalent for any (X, τ) :

(1) $(X, \tau SO)$ is countably compact.

(2) $(X, \tau SO)$ is compact.

(3) (X, τ) is semi-countably compact.

(4) (X, τ) is semi-compact.

(5) Cellular families and nowhere dense subsets are all finite in (X, τ).

PROOF : (2) $\Leftrightarrow$ (4) follows easily from Alexander's subbase lemma. (4) $\Leftrightarrow$ (5) has been shown in a generalized form in Ganster *et al.*[6]. (2) $\Rightarrow$ (1) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (5) Suppose $N \subseteq X$ is a countably infinite and nowhere dense set. Then the countable cover $\{S_n : n \in \omega\}$ where $S_n = (X - N) \cup \{x_n\}$ and $N = \{x_n : n \in \omega\}$ would have no finite subcover. Thus every nowhere dense subset of a space (X, τ) satisfying (3) is necessarily finite. Now suppose $\{G_n : n \in \omega\}$ is a cellular family and let $G = \bigcup \{G_n : n \in \omega\}$ and $\omega = \bigcup \{I_n : n \in \omega\}$ be a partition where each I_n is infinite. Then $G = \bigcup \{U_n : n \in \omega\}$ where $U_n = \bigcup \{G_i : i \in I_n\}$. By the preceding lemma there exists $n_0 \in \omega$ such that

$$\mathrm{cl}G = U_{n_0} \cup \mathrm{cl}\ (\bigcup_{n \neq n_0} U_n).$$

Consequently $\{X - \mathrm{cl}\ G, \mathrm{cl}\ (\bigcup \{U_n : n \neq n_0\})\} \cup \{G_i : i \in I_{n_0}\}$ is a countable semi open cover with no finite subcover. Thus cellular families in a space satisfying (3) have to be finite.

Corollary 6 — Let (X, τ) be a weakly Hausdorff space. Then the following are equivalent :

(1) $(X, \tau SOSO)$ is compact.

(2) $(X, \tau SOSR)$ is compact.

(3) $(X, \tau SO)$ is compact.

(4) X is finite.

PROOF : It suffices to recall the well-known fact that a weakly Hausdorff space is semi-compact iff it is finite.

Proposition 7 — Let (X, τ) be a regular space which is first countable at some non-isolated points. Then the following are equivalent :

(1) $(X, \tau SR)$ is compact.

(2) $(X, \tau SR)$ is countably compact.

(3) X is finite.

PROOF : The space (X, τ) with properties of the hypothesis would not be countably S-closed if it contains infinite number of points by the Corollary 2.5(i) of Dlaska *et al.*[4]. Thus the condition (2) implies easily (3). Other implications are straightforward.

Corollary 7 — Let (X, τ) be a regular, first countable and dense in itself space. Then $(X, \tau SR)$ is compact iff X is finite.

Proposition 8 — If $(X, \tau SR)$ is countably compact, then (X, τ) is countably S-closed and almost e.d.

PROOF : If $(X, \tau SR)$ is countably compact then (X, τ) is evidently countably S-closed since $RC(X, \tau) \subseteq SR(X, \tau) \subseteq \tau SR$. Now suppose that there exists an $U \in RO(X, \tau)$ such that ∂U is infinite. This boundary can be written as a countable partition $\partial U = \bigcup \{A_n : n \in \omega\}$ and the countable cover $\{X - \mathrm{cl}\, U\} \cup \{U \cup A_n : n \in \omega\}$ by semi-regular sets would have then no finite subcover.

Questions

(1) Is the converse of the above statement true, i.e. is $(X, \tau SR)$ countably compact whenever (X, τ) is countably S-closed and almost e.d. ?

(2) If $(X, \tau SR)$ is countably compact, then (X, τ_s) is evidently so since $\tau_s \subseteq \tau SR$. After Singal and Mathur[15] a topological space is called δ-compact if (X, τ_s) is countably compact. Is $(X, \tau SR)$ countably compact whenever (X, τ) is δ-compact and almost e.d. ?

3. The Examples

(1) Let $X = \beta\omega - \{x_0\}$ where $\beta\omega$ is the Stone-Cech compaction of discrete space ω of all finite ordinals and $x_0 \in \beta\omega - \omega$. It is well known that X is countably compact, regular, e.d. but not compact. Thus $(X, \tau SR) \equiv (X, \tau)$ is countably compact but not compact by the Corollaries 2 and 3. Therefore compactness and countable compactness of $(X, \tau SR)$ does not necessarily coincide.

(2) Every e.d. space is evidently almost e.d. and all its points are e.d. points. It is clear that a space is e.d. iff all its points are e.d. points. Now let X be the space defined in above and let X_1 and X_2 denote the two disjoint topological copies of X. For any subset $A \subseteq X$, let A_1 (resp. A_2) denote this set as a subset of X_1 (resp. X_2). Let $X^* = X_1 \cup X_2 \cup \{x_0\}$. We define the basic neighbourhood of any point $x \in X_1$ (resp. $x \in X_2$) as $W_1(x)$ (resp. $W_2(x)$) where $W(x)$ denotes a basic neighbourhood of x in X. Let the basic neighbourhoods of x_0 in X^* be the sets.

$$\{x_0\} \cup (W_{x_0} - \{x_0\})_1 \cup (W_{x_0} - \{x_0\})_2,\ x_0 \in W_{x_0} = \mathrm{int}_{\beta\omega} W_{x_0}.$$

Then it is not difficult to see that X_1 and X_2 are regular open sets in X^* and x_0 is the unique non e.d. point of X^*. Thus X^* is an example of an almost e.d. but not e.d. space. Furthermore since X_1 and X_2 are both topological copy of $\beta\omega - \{x_0\}$ and since; as is well known, $\beta\omega - \omega$ is nowhere dense in $\beta\omega$, the subset $(\beta\omega - (\omega \cup \{x_0\}))_1$ is an example of an uncountable nowhere dense set in X^*.

(3) It is clear that if $(X, \tau SO)$ is Lindelöf then (X, τ) is semi-Lindelöf i.e. every cover of X by semi open sets has a countable subcover. The converse is not necessarily true. Following Kunen[9] an uncountable T_2 space X is called Luzin space if X has at most countably many isolated points and nowhere dense subsets in X are countable. It is well known that in $ZFC + CH$, the real line $\mathbb{R}$ contains a dense Luzin subspace, say (X, τ). Then (X, τ) is semi-Lindelöf by Lemma 1.22 of Kunen[9] but $(X, \tau SO)$ is discrete and not Lindelöf.

(4) Let $\mathcal{F}$ be any free ultrafilter on ω. The single ultrafilter topology on $X = \omega \cup \{\mathcal{F}\}$ is an example of an e.d. Hausdorff space[16] such that the family $SR(X, \tau) = CO(X, \tau)$ is not a topology since for any member $A \in \mathcal{F}$ with infinite elements, the open set $\bigcup \{\{a\} : a \in A\}$ is not closed in X.

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