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THEOREM OF AREOLAR COMPLEX CONTINUED FRACTION ABSOLUTE CONVERGENCE

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In this paper the author proves the theorem of areolar complex continued fraction absolute convergence.

1. INTRODUCTION

It is well known that fractions of form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots \, \cfrac{a_k}{b_k + \cdots}}} \qquad \ldots (1)$$

are called the continued fractions, where coefficients a_i and b_i can be constants, real or complex functions. Components a_i are called the partial numerators, b_i the partial denominators, b_0 the free term and fractions of the form a_i/b_i ith links. Fractions of form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots + \cfrac{a_k}{b_k}}} = \frac{P_k}{Q_k}, \qquad (k = 1, 2, \ldots) \qquad \ldots (2)$$

are called significant fractions for continued fractions (1). For evaluation of numerator

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and denominator of kth significant fraction we use the following recursive formulae

$$P_k = b_k \cdot P_{k-1} + a_k \cdot P_{k-2}$$

$$Q_k = b_k \cdot Q_{k-1} + a_k \cdot Q_{k-2} \quad (k = 1, 2, ...) \qquad ... (3)$$

with initial conditions

$$P_0 = b_0,\ P_{-1} = 1,\ Q_0 = 1,\ Q_{-1} = 0 \qquad ... (4)$$

which can be directly checked (see Skorobogatko[1]).

In his earlier paper the author[2] introduced the notion of 'areolar series' and considered the question of its convergence. The areolar series is complex series

$$\sum_{k=0}^{\infty} \bar{z}^k f_k(z) \qquad ... (5)$$

where $f_k(z)$ are arbitrary analytic functions. The following theorem is proved :

Theorem 1 — Let $W(z, \bar{z})$ be continuous complex function in the domain G which contains the origin and let it have continuous areolar derivatives up to the infinite order in G. By areolar derivative we mean the derivative in the sense of differential operator of Kolossov

$$DW = (u_x' - v_y') + i(u_y' + v_x') = 2W_{\bar{z}}'. \qquad ... (6)$$

Then it may be developed in areolar series of form :

$$W(z, \bar{z}) = \sum_{k=0}^{\infty} \bar{z}^k \cdot f_k(z) \qquad ... (7)$$

in the neighbourhood of the origin D_1 [D_1 is domain of convergence of series (7) and $D_1 \subset G$]. Coefficients $f_k(z)$ are evaluated by formula

$$f_k(z) = \frac{\alpha_0 D^k W}{2^k \cdot k!}. \qquad ... (8)$$

Remark 1 : Operator α_0 determines mapping of set $\mathcal{W}$ of continuous complex functions in G into set $\mathcal{A}$ of all analytic functions in G in the following fashion : by $a = \alpha_0 W$ we mean functions which are got from $W = W(z, \bar{z})$ by substituting $\bar{z}$ by 0 and leaving z unchanged. Indeed, function $a = \alpha_0 W$ does not contain variable $\bar{z}$, which means that it is, in general case, analytic. In a similar way we may define more general operator $\alpha_{g(z)}$ where $\bar{z} = g(z)$ is the equation of some simple smooth closed contour. That operator corresponds to every continuous complex function $W = W(z, \bar{z})$ an analytic function which is got if variable $\bar{z}$ is substituted by $g(z)$ and variable z is left unchanged. Geometrically that means that functions $W(z, \bar{z})$ and $\alpha_{g(z)} W$ have the same limit on closed contour $\bar{z} = g(z)$.

Definition 1 — For areolar series of form (7) it can be determined continued fraction such that decomposition of any of its nth significant fraction in areolar series coincides with initial areolar series conclusive with $\bar{z}^n$. Such continued fraction is called 'correspondent areolar continued fraction' for given areolar series (7). It can be represented in form

$$w_0(z) + \cfrac{1}{w_1(z)\bar{z} + \cfrac{1}{w_2(z)\bar{z} + \cfrac{1}{\ddots + \cfrac{1}{w_k(z)\bar{z} + \ddots}}}}$$

or in condensed form

$$w_0(z) + \frac{1|}{|w_1(z)\bar{z}} + \frac{1|}{|w_2(z)\bar{z}} + \ldots + \frac{1|}{|w_k(z)\bar{z}} + \ldots \qquad \ldots (9)$$

where coefficients $w_0(z)$, $w_1(z)$, ..., $w_k(z)$ ... are detemined by coefficients $f_k(z)$.

In the case that the finite limit value of significant fractions sequence exists, we say that the continued fraction (9) converges and the mentioned limit value represents, in the general case, some nonanalytical function $s(z, \bar{z})$, i.e.

$$s(z, \bar{z}) = \lim_{k \to \infty} \frac{P_k}{Q_k}.$$

2. On α-Interpolation by Areolar Continued Fractions

Definition 2 — Given nonanalytic complex function $W(z, \bar{z})$ and $(n + 1)$ simple smooth closed contours whose equations are $\bar{z} = g_0(z)$, $\bar{z} = g_1(z)$ .... $\bar{z} = g_n(z)$ where all analytic functions $g_i(z)$ $(i = 0, 1, 2, .., n)$ are mutually different. By α-interpolation we mean an approximation of function $W(z, \bar{z})$ by continued fraction $s(z, \bar{z})$ such that the following conditions :

$$\alpha_{g_0(z)} W = \alpha_{g_0(z)} s(z, \bar{z}) = s_0(z), \; \alpha_{g_1(z)} W = \alpha_{g_1(z)} s(z, \bar{z}) = s_1(z), \ldots$$

$$\ldots, \alpha_{g_n(z)} W = \alpha_{g_n(z)} s(z, \bar{z}) = s_n(z) \qquad \ldots (10)$$

hold.

Since the interpolating continued fraction on given contours $\bar{z} = g_i(z)$ $(i = 0, 1, ..., n)$ takes the same limits as the initial function $W = W(z, \bar{z})$, that means that sense of the interpolation is in approximation of one more compound complex function by continued fraction which is in various occasions more convenient for numerical evaluations and computer application. On the other hand, this interpolation may be of great significance in the theory of boundary-value problems for nonanalytic functions.

Remark 2 : From given function $W(z, \overline{z})$, by use of values $g_j(z)$ ($j = 0, 1, 2, ..., n$), we can evaluate needed values $\alpha_{g_0(z)} W$, $\alpha_{g_1(z)} W$ ... $\alpha_{g_n(z)} W$. If, on the contrary, function $W(z, \overline{z})$ is not given, then values $s_0(z), s_1(z), ..., s_n(z)$ must be given in order to construct requested continued fraction.

Interpolating continued fraction in sense of α-interpolation of form (10) form the following sequence of functions

$$F_0(z,\overline{z}) = W(z,\overline{z}), \quad F_1(g_0(z),\overline{z}) = \frac{\overline{z}-g_0(z)}{W(z,\overline{z}) - \alpha_{g_0(z)} W}$$

$$F_2(g_0(z), g_1(z), \overline{z}) = \frac{\overline{z}-g_1(z)}{F_1(g_0(z),\overline{z}) - F_1(g_0(z), g_1(z))}$$

....

$$F_k(g_0(z), g_1(z), ..., g_{k-1}(z), \overline{z}) = \frac{\overline{z}-g_{k-1}(z)}{F_{k-1}(g_0, g_1, ..., g_{k-2}, \overline{z}) - F_{k-1}(g_0, g_1, ..., g_{k-2}, g_{k-1})}$$

The expression

$$F_1(g_0(z), g_1(z)) = \frac{g_1(z)-g_0(z)}{\alpha_{g_1(z)} W - \alpha_{g_0(z)} W}$$

we shall call the first inverse α-areolar difference of the function $W(z, \overline{z})$ for values of arguments $g_0(z)$ and $g_1(z)$. Analogously, the expression $F_2(g_0(z), g_1(z), g_2(z))$ represents the second inverse α-areolar difference and expression $F_n(g_0(z), ..., g_n(z))$ the nth inverse α-areolar difference for values of arguments $g_0(z), g_1(z), .., g_n(z)$. By use of the differences, initial function $W(z, \overline{z})$ may be represented in the following manner:

$$W(z,\overline{z}) = \alpha_{g_0} W + \frac{\overline{z}-g_0}{F_1(g_0,\overline{z})} = \alpha_{g_0} W + \cfrac{\overline{z}-g_0}{F_1(g_0,g_1) + \cfrac{\overline{z}-g_1}{F_2(g_0,g_1,\overline{z})}} = \ldots.$$

$$= \alpha_{g_0} W + \cfrac{\overline{z}-g_0}{F_1(g_0,g_1) + \cfrac{\overline{z}-g_1}{F_2(g_0,g_1,g_2) + \cdots + \cfrac{\overline{z}-g_{n-1}}{F_n(g_0 \cdots g_n) + \cfrac{\overline{z}-g_n}{F_{n+1}(g_0 \cdots g_n, \overline{z})}}}} \quad \ldots (11)$$

If one of the partial numerators of continued fraction (11) is equal to zero, then that link together with all links that follow it can be rejected. Substituting in (11)$\bar{z}$ by $g_0(z), g_1(z), \ldots, g_n(z)$ successively, we get on the right-hand side $\alpha_{g_0(z)} W = s_0(z)$, $\alpha_{g_1(z)} W = s_1(z) \ldots \alpha_{g_n(z)} W = s_n(z)$. From the fact it follows that if we reject last link in (11) we will get the rational function which on contours $\bar{z} = g_0(z)$, $\bar{z} = g_1(z) \ldots$ $\bar{z} = g_n(z)$ takes the same limit as function $W(z, \bar{z})$ and which approximates it.

3. Some Auxiliary Theorems

Definition 3 — The continued fraction

$$w_0(z) + \cfrac{1}{w_1(z)\cdot\bar{z} + \cfrac{1}{w_2(z)\cdot\bar{z} + \cfrac{1}{\ddots \cfrac{1}{w_k(z)\cdot\bar{z} + \ddots}}}} \qquad \ldots (12)$$

absolutely converges in certain finite closed domain G if the continued fraction

$$|w_0(z)| + \cfrac{1}{|w_1(z)\cdot\bar{z}| + \cfrac{1}{|w_2(z)\cdot\bar{z}| + \cfrac{1}{\ddots}}} \qquad \ldots (13)$$

converges in this domain.

Theorem 2 — Assume that the sequence of functions $w_0(z).\, w_1(z).\, \ldots\, w_n(z)$ analytical in certain finite closed domain G is given. Then for the continued fraction (12) the formulae

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = \frac{(-1)^{k+1}}{Q_k \cdot Q_{k-1}} \qquad \ldots (14)$$

$$\frac{P_k}{Q_k} - \frac{P_{k-2}}{Q_{k-2}} = \frac{(-1)^{k}}{Q_k \cdot Q_{k-2}} \qquad \ldots (15)$$

are valid.

PROOF : On the base of the formulae (3) we have

$$P_k = b_k P_{k-1} + a_k P_{k-2} = w_k(z)\overline{z} \cdot P_{k-1} + P_{k-2}$$

$$Q_k = b_k Q_{k-1} + a_k Q_{k-2} = w_k(z)\overline{z} \cdot Q_{k-1} + Q_{k-2}.$$

If we replace these values in (14) we obtain

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = \frac{P_k Q_{k-1} - Q_k P_{k-1}}{Q_k \cdot Q_{k-1}}$$

$$= \frac{(w_k(z) \cdot \overline{z} \cdot P_{k-1} + P_{k-2}) \cdot Q_{k-1} - (w_k(z) \cdot \overline{z} \cdot Q_{k-1} + Q_{k-2}) P_{k-1}}{Q_k \cdot Q_{k-1}}.$$

$$= (-1)\frac{P_{k-1} Q_{k-2} - Q_{k-1} P_{k-2}}{Q_k \cdot Q_{k-1}} = \dots = \frac{(-1)^{k+1}}{Q_k \cdot Q_{k-1}}.$$

In the same way we show that (15) is also valid.

Theorem 3 — Consider the continued fraction (13) where $w_i(z)$ ($i = 0, 1, \dots, n$) are arbitrary analytical functions in certain finite closed domain G. Then for the valid fractions of the continued fraction (13) in the domain G the following inequalities

$$\frac{P_1}{Q_1} > \frac{P_3}{Q_3} > \frac{P_5}{Q_5} > \dots > \frac{P_{2i+1}}{Q_{2i+1}} > \dots. \qquad \dots (16)$$

$$\frac{P_0}{Q_0} < \frac{P_2}{Q_2} < \frac{P_4}{Q_4} < \dots < \frac{P_{2k}}{Q_{2k}} < \dots. \qquad \dots (17)$$

$$\frac{P_{2i-1}}{Q_{2i-1}} > \frac{P_{2k}}{Q_{2k}}, \ (i = 1, 2, \dots, k = 0, 1 \dots) \qquad \dots (18)$$

are valid.

PROOF : On the base of the formula (15)

$$\frac{P_{2i+1}}{Q_{2i+1}} - \frac{P_{2i-1}}{Q_{2i-1}} = \frac{(-1)^{2i+1}}{Q_{2i+1} \cdot Q_{2i-1}} < 0$$

$$\frac{P_{2k}}{Q_{2k}} - \frac{P_{2k-2}}{Q_{2k-2}} = \frac{(-1)^{2k}}{Q_{2k} \cdot Q_{2k-2}} > 0 .$$

We can notice directly that the third inequality is valid, too.

Consequence : The necessary and sufficient criterion for the continued fraction convergence (13) is the condition

$$\lim_{p \to \infty} \left(\frac{P_{2p+1}}{Q_{2p+1}} - \frac{P_{2p}}{Q_{2p}} \right) = 0. \qquad \dots (19)$$

4. Theorem of Areolar Complex Continued Fraction Absolute Convergence

Theorem 4 — Assume that the sequence of functions $w_0(z), w_1(z), ..., w_n(z)$ analytical in certain finite closed domain G is given. Consider the continued fraction (13) that could be written also in the abridged form

$$| w_0(z) | + \sum_{i=1}^{\infty} \frac{1 |}{| | w_i(z) \bar{z} |} . \qquad ... (20)$$

The necessary and sufficient condition for the continued fraction convergence in the domain G is that the series $\sum_{i=1}^{\infty} |w_i(z)|$ diverges in the same domain.

Sufficiency of condition : Let us estimate the denominators of the valid fractions Q_{2n+1} and Q_{2n} from the lower side. Using the formulae (3) and (4) we have

$$Q_1 = |w_1(z)| \; |\bar{z}| , \; Q_2 = | w_1(z) | \; | w_2(z) | \; | \bar{z} |^2 + 1$$

$$... Q_{2n} = | w_{2n}(z) | \; | \bar{z} | \; Q_{2n-1} + Q_{2n-2}$$

$$Q_{2n+1} = | w_{2n+1}(z) | \; | \bar{z} | \; Q_{2n} + Q_{2n-1}.$$

It is obvious that

$$Q_2 > 1, \; Q_4 = |w_4(z)| \; |\bar{z}| \; Q_3 + Q_2 > 1 \; ... \; Q_{2n} > 1.$$

From the other side

$$Q_3 = |w_3(z)| \; |\bar{z}| \; Q_2 + Q_1 > Q_1 \; .... \; Q_{2n+1} > Q_1.$$

That means that

$$Q_{2n} > Q_{2n-2} + |w_1(z)| \; |w_{2n}(z)| \; |\bar{z}|^2$$

$$Q_{2n+1} > Q_{2n-1} + |w_{2n+1}(z)| \; |\bar{z}| . \qquad ... (21)$$

If we successively apply these inequalities, we will finally obtain the estimates

$$Q_{2n} > |w_1(z)| \; |w_{2n}(z)| \; |\bar{z}|^2 + |w_1(z)| \; |w_{2n-2}(z)| \; |\bar{z}|^2 + Q_{2n-4}$$

$$> ... > |w_1(z)| \; (|w_2(z)| + ... + |w_{2n}(z)|) \; |\bar{z}|^2 \qquad ... (22)$$

$$Q_{2n+1} > |w_{2n+1}(z)| \; |\bar{z}| + |w_{2n-1}(z)| \; |\bar{z}| + Q_{2n-3}$$

$$> ... > (|w_1(z)| + |w_3(z)| + ... + |w_{2n+1}(z)|) \; |\bar{z}| . \qquad ... (23)$$

If the series $\sum_{n=1}^{\infty} |w_n(z)|$ diverges in domain G, then at least one of the series

$$\sum_{n=1}^{\infty} |w_{2n}(z)| \text{ or } \sum_{n=1}^{\infty} |w_{2n+1}(z)|$$

diverges in this domain. But then on the base of (22), (23) and (14)

$$\lim_{n \to \infty} \left(\frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} \right) = 0.$$

Necessity of condition : Let us estimate the denominators of the valid fractions from the upper side. We have that

$$Q_1 = |w_1(z)|\ |\bar{z}| < 1 + |w_1(z)|\ |\bar{z}|$$

$$Q_2 = |w_1(z)|\ |w_2(z)|\ |\bar{z}|^2 + 1 < (1 + |w_1(z)|\ |\bar{z}|)(1 + |w_2(z)|\ |\bar{z}|).$$

If on the base of the mathematical induction method we assume that

$$Q_k < (1 + |w_1(z)|\ |\bar{z}|)(1 + |w_2(z)|\ |\bar{z}|) \dots (1 + |w_k(z)|\ |\bar{z}|)$$

$$k = 1, 2, \dots, n-1$$

then

$$Q_n = |w_n(z)|\ |\bar{z}|\ Q_{n-1} + Q_{n-2} < (1 + |w_1(z)|\ |\bar{z}|)(1 + |w_2(z)|\ |\bar{z}|) \dots$$

$$(1 + |w_{n-2}(z)|\ |\bar{z}|)[\,|w_n(z)|\ |\bar{z}|\ (1 + |w_{n-1}(z)|\ |\bar{z}|) + 1]$$

$$< (1 + |w_1(z)|\ |\bar{z}|)(1 + |w_2(z)|\ |\bar{z}|) \dots (1 + |w_n(z)|\ |\bar{z}|).$$

Since for the arbitrary positive number x the inequality $1 + x < e^x$ is valid, then

$$Q_n < e^{|\bar{z}|\,(|w_1(z)| + |w_2(z)| + \dots + |w_n(z)| + \dots)}$$

If the series $\sum_{n=1}^{\infty} |w_n(z)|$ converges and if its sum in the finite closed domain G is limited by the constant C, then the inequality

$$\left| \frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} \right| > \frac{1}{e^{2\alpha C}},$$

where $|\bar{z}| < \alpha$ in G, is valid for arbitrary n. But then the continued fraction (20) should diverge.

Thus, the theorem is completely proved.

5. Conclusion

Complex function $W(z, \bar{z})$ may be approximated either by interpolating continued fraction or by areolar polynomial and both of these interpolations may have their advantages. Interpolation by continued fraction is specially effective in case of rational functions because they can be written in the form of finite areolar continued fraction in comparison to representation by areolar series which gives an infinite sum. It further means that α-interpolating areolar continued fraction can always be identified with rational complex function $w(z, \bar{z})$ itself if sufficient boundary conditions of type (10) are given. In case of areolar interpolating polynomial there is an interpolation error whenever rational function does not degenerate into polynomial. On the other hand, continued fractions are convenient in case of application of computers, which is shown in Hovanskii[3]. All these facts motive for the development of the theory of continued fractions and its application in real, complex and numerical analysis, algebra and other mathematical disciplines.

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