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A NOTE ON HYPERCYCLIC OPERATORS ON THE SPACE OF ENTIRE SEQUENCES

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In this short paper, the hypercyclicity (density of an orbit) of certain weighted backward shift operators on the space of entire sequences is proved. Our result generalizes MacLane's Theorem, which asserts the hypercyclicity of differentiation operator, acting on the space of entire functions.

1. INTRODUCTION

In this paper by an 'operator' we shall mean a linear map from a linear space into itself. Let T be an operator on a linear topological space X. The orbit of the vector $x \in X$ under the operator T is denoted and defined as $\mathrm{Orb}(T, x) = \{T^n x : n \geq 0\}$. A vector $x \in X$ is called 'hypercyclic' under the operator T if the orbit of x under T is dense in X. The operator T is said to be 'hypercyclic on X' if there is at least one vector x in X whose orbit under T is dense in X.

A hypercyclic vector x for T is also called a universal vector in X because every vector in X is a limit point of the orbit of x. Thus the study of hypercyclic vectors seems to be of interest from the point of view of approximation theory.

The credit for obtaining the first hypercyclicity theorem goes to Birkhoff[1]. He proved that translation operators other than the identity operator acting on $H(\mathbb{C})$, the space of entire functions of one complex variable, are hypercyclic. Later MacLane[6], showed that differentiation operator acting on $H(\mathbb{C})$ is hypercyclic. The study of hypercyclicity for Banach and Hilbert space operators originated with Rolewicz[7], who showed that the scalar multiple of the backward shift operator on sequence spaces l^p, $(1 \leq p < \infty)$, c_0, by a scalar $a > 1$, has a hypercyclic vector. Rolewicz's result inspired Kitai[5] and Gethener and Shapiro[3] to find independently the following sufficient condition for hypercyclicity of a continuous linear operator on a separable complete linear metric space, which we shall use to prove our main result in this paper.

Proposition 1.1 — Suppose T is a continuous linear operator on a separable, complete linear metric space X for which the sequence of powers $\{T^n : n \geq 0\}$ tends pointwise to zero on a dense subset of X. If there is a (possibly different) dense subset Y of X, and a map (possibly non-linear, possibly discontinuous) $S : Y \to Y$ such that TS = identity on Y, and $\{S^n : n \geq 0\}$ tends pointwise to zero on Y, then the operator T is hypercyclic.

The version stated here is Corollary 1.5 of Godefroy and Shapiro[4].

2. Hypercyclic Operators on Space of Entire Sequences

Ganapathy Iyer[2] has introduced a metric on $H(\mathbb{C})$ by regarding $H(\mathbb{C})$ as a special class of sequences of complex numbers. We shall call this sequence space the space of 'entire sequences'. He (Ganapathy Iyer[2], Theorem 3) further proved that the topology generated on $H(\mathbb{C})$ by this metric is equivalent to the usual topology of $H(\mathbb{C})$ (i.e. the topology of uniform convergence on compact subsets of the plane). To keep this paper self contained, we shall include the following definitions.

Definition 2.1 — It is well known that $f(z) = \sum_{n=0}^{\infty} a_n z^n$, a_n's are scalars, is an entire function if and only if $\lim_{n \to \infty} | a_n |^{1/n} = 0$. The sequence $\{a_n\}$ is called an entire sequence. If $g(z) = \sum_{n=0}^{\infty} b_n z^n$, is also an entire function, then

$$d(f, g) = \sup \left\{ | a_0 - b_0 |, | a_n - b_n |^{1/n} : n \geq 1 \right\} \qquad \dots (1)$$

is a metric on the class of all entire sequences. Ganapathy Iyer[2] (Theorem 1) showed that $H(\mathbb{C})$ is a separable, complete linear space with respect to the topology induced by the metric defined in (1).

Definition 2.2 (weighted forward/backward shift) — Let $f \in H(\mathbb{C})$. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$. The weighted backward shift operator B acting on $H(\mathbb{C})$ with weights $\{W_n : n \geq 1\}$ is defined as

$$Bf(z) = \sum_{n=1}^{\infty} W_n a_n z^n.$$

That is, $B(a_0, a_1, a_2, \dots, a_n \dots) = (W_1 a_1, W_2 a_2, \dots, W_n a_n, \dots)$.

Similarly, the 'weighted forward shift' operator S acting on $H(\mathbb{C})$ with weights $\{W_n : n \geq 1\}$ is defined as

$$Sf(z) = \sum_{n=0}^{\infty} W_{n+1} a_n z^n.$$

That is, $S(a_0, a_1, a_2, ..., a_n, \; ...) = (0, \; W_1 a_0, W_2 a_1, ..., W_{n+1} a_n, \; ...)$. The weights $\{W_n : n \geq 1\}$ are chosen in such a way that B and S are operators on $H(\mathbb{C})$.

Now we are ready to prove the main result of this paper.

The proof of this result follows after some remarks.

Theorem 2.2 — The weighted backward shift operator B acting on $H(\mathbb{C})$ with non-zero weights $\{W_n : n \geq 1\}$ such that $W_n \to \infty$ as $n \to \infty$ and $\omega = \sup\{| W_n |^{1/n} : n \geq 1\} < \infty$, is hypercyclic.

Remarks 2.3 : (1) The above theorem can be proved for $H(Y) = \{y = (y_0, y_1, ..., y_n, ...), y_i \in Y : \lim_{n \to \infty} \| y_n \|_y^{1/n} = 0.\}$, the space of entire sequences in Y where Y is a separable Banach space, using the topology generated by the metric

$$d(x, \; y) = \sup\{\| x_0 - y_0 \|_y, \| x_n - y_n \|_y^{1/n} : n \geq 1\},$$

$$x = (x_0, x_1, ..., x_n,) \in H(Y).$$

(2) In the above theorem, if we take $W_n = n, n \geq 1$, we will get MacLane's Theorem[6] on the hypercyclicity of differentiation operator on the space of entire functions.

(3) Godefroy and Shapiro[4] proved that no scalar multiple of the backward shift on $H(\mathbb{C})$ is hypercyclic. Hence the condition, $W_n \to \infty$ as $n \to \infty$ can't be omitted from the hypotheses of the above theorem.

Proof of Theorem 2.2 — Throughout the course of the proof of this theorem, we shall assume that $H(\mathbb{C})$ is equipped with the sequential topology, generated by the metric (1).

Let f, $g \in H(\mathbb{C})$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

where a_n's and b_n's are scalars. By the given hypotheses

$$Bf(z) = \sum_{n=1}^{\infty} W_n a_n z^n.$$

Then $d(Bf, Bg) \leq \omega \, d(f, g)$. This implies that B is continuous on $H(\mathbb{C})$.

The set M which consists of all polynomials with complex rational coefficients is dense in $H(\mathbb{C})$ (Ganapathy Iyer[2], Theorem 1).

Clearly, $B^n \to 0$ on M.

Let S be the weighted forward shift operator acting on M with weights $\{1/W_n : n \geq 1\}$. Then BS = identity on M.

Let $h \in M$. Then $h(z) = \sum_{n=0}^{\infty} c_n z^n$ for some $m > 0$.

Now, for some $k > 0$

$$S^k h(z) = \sum_{n=0}^{m} \frac{c_n}{W_{n+1} W_{n+2} \cdots W_{n+k}} z^n.$$

Then

$$d(S^k h, 0) = \sup\{(a_{i,k})^{1/(k+i)} : 0 \le i \le m\} \qquad \dots (2)$$

where, $a_{i,k} = |c_i|/|W_{i+1} W_{i+2} \dots W_{i+k}|$, $0 \le i \le m$. Clearly, $a_{i,k} > 0$, $0 \le i \le m$.
Also, $\lim_{k \to \infty} (a_{i,k+1}/a_{i,k}) = \lim_{k \to \infty} (1/|W_{i+k+1}|) = 0.$

Thus $0 = \lim_{k \to \infty} (a_{i,k+1}/a_{i,k}) = \lim_{k \to \infty} (a_{i,k})^{1/(k+i)}$. (Rudin[8], Theorem 3.37). This, along with eqn. (2) implies that $S^k f \to 0$ in $H(\mathbb{C})$ for every $f \in M$. Thus by Proposition 1.1, the operator B is hypercyclic on $H(\mathbb{C})$. Hence the theorem.

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