Indian J. pure appl. Math., 25(12): 1275-1280, December 1994

ON A SPECIAL FORM OF THE *hv*-CURVATURE TENSOR OF BERWALD'S CONNECTION $B\Gamma$ OF FINSLER SPACE

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(*Received* 8 *November* 1993; *after revision* 29 *July* 1994; *accepted* 1 *September* 1994)

A special form of the *hv*-curvature tensor of Berwald's connection $B\Gamma$ of a Finsler space is proposed in this paper. It arises from the theory of two-dimensional Finsler space and it has been shown that Douglas tensor D vanishes of such a Finsler space.

1. INTRODUCTION

Let F^n be an n-dimensional Finsler space equipped with metric function $L(x, y)$. The metric tensor and angular metric tensor of F^n are given by

$$g_{ij} = \frac{1}{2} \dot{\partial}_i \; \dot{\partial}_j \; L^2$$

$$h_{ij} = g_{ij} - l_i \, l_j$$

where $l_i = \dot{\partial}_i L = \dfrac{\partial L}{\partial y^i}$ is the unit vector along the element of support y^i. The Cartan's C tensor C_{ijk} is defined as

$$C_{ijk} = \frac{1}{2} \dot{\partial}_k \; g_{ij}$$

which is completely symmetric and indicatory tensor[4]. For a given Finsler connection in F^n there exist two kinds of covariant differentiations called h- and v-covariant differentiations. In the present paper we are concerned with the Berwald's connection $B\Gamma = \left(G^i_{jk} \, , \; G^i_j \, , \; 0 \right)$ with respect to which the h- and v-covariant differentiations of a vector field X^i are given by

$$X^i_{(j)} = \partial_j X^i - (\partial_k X^i)\, G^k_j + X^h G^i_{hj} \qquad \text{... (1.1)}$$

$$X^{\ i}_{\cdot j} = \dot{\partial}_j X^i \qquad \text{... (1.2)}$$

where $\partial_j = \dfrac{\partial}{\partial X^j}$.

Recently a Finsler connection $P\Gamma = (P^i_{jk}, G^i_j, 0)$ of a Finsler space F^n has been introduced (see Matsumoto[5]) which is called the projective connection of F^n. Here

$$P^i_{jk} = G^i_{jk} - G_{jk}\, y^i/(n+1), \qquad \text{... (1.3)}$$

$$G_{jk} = G^i_{jki}, \qquad \text{... (1.4)}$$

$$G^i_{hjk} = \partial_h\, G^i_{jk}. \qquad \text{... (1.5)}$$

G^i_{hjk} and G_{jk} are called the hv-curvature tensor and hv-Ricci tensor of $B\Gamma$ respectively.

We have two essential projective invariants, one being the Weyl curvature tensor W and the other the Douglas tensor D. From the standpoint of Matsumoto's theory of general Finsler connections[4], while W is of h-type, D is of hv-type with respect to $B\Gamma = (\Gamma^h, \Gamma^v)$. The vanishing of Douglas tensor D must show some special relation between Γ^h, Γ^v.

In this paper we are concerned with a special form of the hv-curvature tensor of $B\Gamma$ given by

$$G^i_{hjk} = s_{(hjk)}\, \{(y^i a_{;i} + \lambda\, h^i_h)\, h_{jk}\} \qquad \text{... (1.6)}$$

where $s_{(hjk)}\ \{\ \}$ denotes the cyclic permutation of h, j, k and summation. Here a_h are components of some covariant vector positively homogeneous of degree -2 in y^i and λ is a scalar function positively homogeneous of degree -1 in y^i. This form of G^i_{hjk} arises from the form of hv-curvature tensor of two-dimensional Finsler space.

2. Special Finsler Spaces with the Form (1.6)

Since $G^i_{hjk}\, y^h = G^i_{0jk} = 0$ and h_{ij} is an indicatory tensor, from (1.6) we get $a_0 = 0$. Again from (1.4) and (1.6) we get

$$G_{hj} = (n+1)\, \lambda\, h_{hj}. \qquad \text{... (2.1)}$$

A Landsberg space is characterized by the relation $g_{ij(k)} = 0$ which is equivalent to $C_{ijk\,|\,0} = 0$ (Matsumoto[4], p.162, Numata[6]). This condition is also equivalent to $G^i_{hjk}\, y_i = 0$ which can be proved as follows.

From (18.2) and (18.6) in Matsumoto[4] we have

(i) $G^i_{hjk} = P^i_{hjk} + C^i_{hk\,|\,j} - C^i_{hr}\, P^r_{jk} + C^i_{jh\,|\,0\cdot k}$

where $C^i_{jh|0} = P^i_{jh}$. From "Landsberg space" we have $P^i_{hjk} = 0$ and $P^i_{jk} = 0$, and thus from (i)

(ii) $G^i_{hjk} = C^i_{hk|j}$.

Transvecting (ii) with y_i, we have $y_i G^i_{hjk} = 0$.

Conversely if we assume $y_i G^i_{hjk} = 0$ we have from (i)

(iii) $y_i P^i_{hjk} + y_i P^i_{jh \cdot k} = 0$.

As from (17.23) in Matsumoto[4] we have

$$y_i P^i_{hjk} = y^i P_{hijk} = -y^i P_{ihjk} = -C_{hjk|0}$$

and
$$y_i P^i_{jh \cdot k} = (y_i C^i_{jh|0})_{\cdot k} - y_{i \cdot k} C^i_{jh|0} = -C_{jkh|0}.$$

We have from (iii) $C_{hjk|0} = 0$ as $y_{i \cdot k} = g_{ik}$. In view of (1.6) and $G^i_{hjk} y_i = 0$ we get $S_{(hjk)} \{a_h h_{jk}\} = 0$ which yields $a_i = 0$. Thus we have the following :

Theorem 2.1 — If the hv-curvature tensor of a Landsberg space is of the form (1.6) then $a_i = 0$ and $G^i_{hjk} = s_{(hjk)} \{\lambda\, h^i_h\, h_{jk}\}$.

In the theory of two-dimensional Finsler space we can refer to the Berwald's frame (l^i, m^i) (Matsumoto[4]) where $l^i = y^i/L$ and m^i is the unit vector orthogonal to l^i. The angular metric tensor h_{ij} and hv-curvature tensor G^h_{ijk} of $B\Gamma$ are written in the form (Matsumoto[4])

$$h_{ij} = m_i m_j \qquad \text{... (2.2)}$$

$$L\, G^i_{hjk} = (-2I_{(1)}\, l^i + I_2\, m^i)\, m_h\, m_j\, m_k \qquad \text{... (2.3)}$$

where I is the main scalar and $I_2 = I_{(2)} + I_{(1) \cdot 2}$.

From (2.3) and (2.2) it follows that G^i_{hjk} can be written in the form (1.6) where $a_i = -\dfrac{2I_{(1)}}{3L^2} m_i$ and $\lambda = \dfrac{I_2}{3L}$, thus we have :

Theorem 2.2 — The hv-curvature tensor of $B\Gamma$ of any two-dimensional Finsler space can be written in the form of (1.6).

Next we shall discuss the three-dimensional Finsler space F^3. Matsumoto[2] developed the theory of three-dimensional Finsler space referring to the orthogonal frame $e^i_{(\alpha)}$, $\alpha = 0, 1, 2$, called the Moor frame (Matsumoto[4]). With reference to this frame, the angular metric tensor h_{ij} is written as

$$h_{ij} = (\delta_{\alpha\beta} - \delta_{0\alpha}\, \delta_{0\beta})\, e_{(\alpha)i}\, e_{(\beta)j}. \qquad \text{... (2.4)}$$

If $G_{\alpha\beta\gamma\delta}$ and a_α are scalar components of G^i_{hjk} and a_i respectively i.e.

$$G^i_{hjk} = G_{\alpha\beta\gamma\delta}\, e^i_{(\alpha)}\, e_{(\beta)h}\, e_{(\gamma)j}\, e_{(\delta)k} \qquad \text{... (2.5)}$$

and $$a_i = a_\alpha e_{(\alpha)i} \qquad \text{... (2.6)}$$

then from (1.6) and (2.4) we get

$$G_{\alpha\beta\gamma\delta} = s_{(\beta\gamma\delta)} (L\delta_{0\alpha} a_\beta + \lambda \delta_{\alpha\beta} - \lambda\delta_{0\alpha} \delta_{0\beta}) (\delta_{\gamma\delta} - \delta_{0\gamma} \delta_{0\delta}). \qquad \text{... (2.7)}$$

Since G^i_{hjk} is symmetric in the indices h, j, k and $G^i_{0jk} = 0$, it follows that the scalar components $G_{\alpha 0\gamma\delta}$, $G_{\alpha\beta 0\delta}$, $G_{\alpha\beta\gamma 0}$ vanish identically and from (2.7) we get

(a) $G_{0111} = 3La_1$, $G_{0112} = La_2$, $G_{0122} = La_1$, $G_{0222} = 3La_2$

(b) $G_{1111} = 3\lambda$, $G_{1112} = 0 = G_{1222}$, $G_{1122} = \lambda$

(c) $G_{2111} = 0 = G_{2122}$, $G_{2112} = \lambda$, $G_{2222} = 3\lambda$. .... (2.8)

From (2.8) we have the following :

Theorem 2.3 — In a three-dimensional Finsler space if *hv*-curvature tensor of $B\Gamma$ is of the form (1.6) then the scalar components $G_{\alpha\beta\gamma\delta}$ of *hv*-curvature tensor G_{hjk} satisfy the following identities

$$G_{\alpha 0 \beta\gamma} = G_{1112} = G_{1222} = G_{2111} = G_{2122} = 0$$

$$G_{0111} = 3G_{0122}, \quad G_{0222} = 3G_{0112}$$

and

$$G_{1111} = G_{2222} = 3G_{1122} = 3G_{2112}.$$

3. The Douglas Tensor of F^n

The $(v)hv$-curvature tensor U^i_{jk} of projective connection $P\Gamma$ is given by (Matsumoto[5])

$$U^i_{jk} = \frac{1}{n+1} G_{jk} y^i , \qquad \text{... (3.1)}$$

which in view of (2.1) gives

$$U^i_{jk} = \lambda h_{jk} y^i. \qquad \text{... (3.2)}$$

The *hv*-curvature tensor of $P\Gamma$ is defined (Matsumoto[5]) as

$$U^i_{hjk} = \dot\partial_k P^i_{hj}$$

which in view of (1.3) gives

$$U^i_{hjk} = G^i_{hjk} - (G_{hj\cdot k} y^i + G_{hj} \delta^i_k)/(n+1). \qquad \text{... (3.3)}$$

Since

$$h_{ij\cdot k} = 2C_{ijk} - L^{-1} (h_{ik} l_j + h_{jk} l_i), \qquad \text{... (3.4)}$$

we have from (2.1)

$$G_{hj \cdot k} = (n+1)\,[\lambda_k\, h_{hj} + \lambda\{2C_{hjk} - L^{-1}\,(l_h\, h_{jk} + l_j\, h_{hk})\}] \qquad \text{... (3.5)}$$

where $\lambda_k = \dot{\partial}_k\, \lambda$.

Now we consider F^n satisfying (1.6). Then from (1.6), (3.3) and (3.5) we get

$$\begin{aligned} U^i_{hjk} = {} & s_{(hjk)}\,\{(y^i\, a_h + \lambda\, h^i_h)\, h_{jk}\} \\ & - [\lambda_k\, h_{hj} + \lambda\ \{2C_{hjk} - L^{-1}\,(l_h\, h_{jk} + l_j\, h_{hk})\}]\ y_i \\ & - \lambda\, h_{hj}\, \delta^i_k. \end{aligned} \qquad \text{... (3.6)}$$

Since the hv-Ricci tensor of $P\Gamma$ is given by (Matsumoto[5])

$$U_{ij} = \frac{2G_{ij}}{n+1},$$

from (2.1) we get

$$U_{ij} = 2\lambda\, h_{ij}. \qquad \text{... (3.7)}$$

The Douglas tensor D^i_{hjk} is given by (Matsumoto[4])

$$D^i_{hjk} = U^i_{hjk} - (U_{jk}\, \delta^i_h + U_{hk}\, \delta^i_j)/2. \qquad \text{... (3.8)}$$

In view of (3.6), (3.7) and (3.8), the Douglas tensor D^i_{hjk} of F^n may be written as

$$\begin{aligned} D^i_{hjk} = y^i\ & \{a_h\, h_{jk} + a_j\, h_{kh} + a_k\, h_{jh} - 2\lambda\, C_{hjk} \\ & - (\lambda_k + L^{-1}\,\lambda\, l_k)\, h_{hj}\}. \end{aligned} \qquad \text{... (3.9)}$$

Since D^i_{hjk} is symmetric in j and k, from (3.9) we get

$$(\lambda_k + L^{-1}\,\lambda\, l_k)\, h_{hj} = (\lambda_j + L^{-1}\,\lambda\, l_j)\, h_{hk}. \qquad \text{... (3.10)}$$

Contracting (3.10) with g^{hj} we get

$$\lambda_k + L^{-1}\,\lambda\, l_k = 0 \qquad \text{... (3.11)}$$

if $n > 2$, which shows that λL does not depend on y^i.

Then (3.9) reduces to

$$D^i_{hjk} = y^i\ E_{hjk},$$

$$E_{hjk} = S_{(h,j,k)}\ \{a_h\, h_{jk}\} - 2\ \lambda\, C_{hjk}. \qquad \text{... (3.12)}$$

The covariant differentiation of (3.12) gives

$$D^i_{hjk \cdot m} = \delta^i_m\, E_{hjk} + y^i\, E_{hjk \cdot m}. \qquad \text{... (3.13)}$$

Since E_{hjk} are positively homogeneous of degree -2 in y^i, (3.13) yields $D^i_{hjk\cdot i} = (n-2)\,E_{hjk}$. Then the identity $D^i_{hjk\cdot i} = 0$ gives $E_{hjk} = 0$ (Matsumoto[4]), provided $n > 2$. Consequently we have

(1) If $\lambda = 0$ identically, then we have $a_h = 0$ and $G^i_{hjk} = 0$, hence F^n is a Berwald space.

(2) If $\lambda \neq 0$, then F^n is a C-reducible Finsler space with $a_h = 2\lambda c_h/(n+1)$ (Matsumoto[1], Matsumoto and Hojo[3]) and (3.12) shows $D^i_{hjk} = 0$.

Summarizing up we have :

Theorem 3.1 — Suppose that the hv-curvature tensor of the Berwald connection of F^n $(n > 2)$ be of the form (1.6). Then λL does not depend on y^i and

(1) if $\lambda = 0$ identically, then F^n is a Berwald space,

(2) if $\lambda \neq 0$, then F^n is a C-reducible Finsler space with $a_i = 2\lambda C_i/(n+1)$ and its Douglas tensor vanishes.

ACKNOWLEDGEMENT

The authors wish to express their sincere gratitude to the referee for many valuable suggestions and comments.

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