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SELF-GRAVITATING INSTABILITY OF A ROTATING STREAMING FLUID JET

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The self-gravitating instability of non-stationary rotating fluid in the axial direction has been studied. A general dispersion relation describing the stability chracters of that cylindrical fluid jet is derived and discussed analytically and numerically. The jet is purely stable in the non-axisymmetric perturbations while in the axisymmetric mode it is unstable for small wavenumber. The streaming has the effect of reducing the stable states not only in the axisymmetric mode but also in those of non-axisymmetric. The rotation has strong stabilizing influence, so the gravitational streaming instability is modified.

INTRODUCTION

The interest in the instability and dynamic oscillations of circular columns systems have been revived in recent decades by certain physical problems in astrophysics domain. Chandrasekhar and Fermi[1] were the first to study the axisymmetric instability of a self-gravitating jet which has a satisfactory correlation with the oscillation of stars, see Robe[2, 3]. Later Chandrasekhar[4] studied this problem using the normal mode analysis for all possible axisymmetric and non-axisymmetric perturbation modes. The interaction of magnetic fields with rotation is discussed for that self-gravitating model by Pits and Tayler[5]. Recently Radwan and Elazab[6] have investigated the stability of a liquid cylinder in an infinite liquid under the influence of self-gravitating force. They[6] have shown that the ratio of densities is very important for analysing such a problem in particular for $m = 0$ and $m = 1$, where m is the transverse wavenumber.

The purpose of the present work is to investigate the stability of a streaming circular fluid column (radius R_0) under the influence of inertia, self-gravitating, and rotating forces. These forces in addition to the pressure gradient force are acting on the fluid.

2. FORMULATION OF THE PROBLEM

We consider an infinite circular cylinder of fluid (density ρ) streams with velocity $\underline{u}$ (= (0, 0, U)) and rotating with angular velocity $\underline{\Omega}$ (= (0, 0, Ω)). The components of $\underline{u}$ and $\underline{\Omega}$ are considered along the utilizing cylindrical polar coordinates (r, ϕ, z) system with the z-axis coinciding with the axis of the fluid cylinder. The fluid acting upon the self-gravitating, inertia, rotating, and pressure gradient forces. The surrounding medium of the fluid cylinder acting upon the self-gravitating force only.

The basic equations required for investigating the problem under consideration are the gravitohydrodynamic equation of motion in a vector form, continuity equation expressing the conservation of mass, the Poisson's equation satisfying the gravitational field interior the fluid, and the Laplace's equation satisfying the gravitational potential of the medium surrounding the fluid cylinder. Under the present situation these equations can be written in the form

$$d\underline{u}/dt = -\underline{\nabla}\pi + 2(\underline{u} \wedge \underline{\Omega}) \qquad \dots (1)$$

$$\pi = p/\rho - V^i - \frac{1}{2}|\underline{\Omega} \wedge \underline{r}|^2 \qquad \dots (2)$$

$$\nabla \cdot \underline{u} = 0 \qquad \dots (3)$$

$$\nabla^2 V^i = -4\pi G\rho \qquad \dots (4)$$

$$\nabla^2 V^0 = 0. \qquad \dots (5)$$

Here ρ, $\underline{u}$, and p are the fluid mass density, velocity vector and kinetic pressure; G is the gravitational constant, V^i is the gravitational potential interior the fluid cylinder and idem V^0 exterior the fluid cylinder, where the superscripts i and 0 indicate interior and exterior the fluid cylinder.

In the unperturbed state the system of the basic equations (1)-(5) takes the form

$$\nabla \pi_0 = 0, \; \pi_0 = p_0/\rho - \frac{1}{2}|\Omega \wedge r|^2 - V_0^i \qquad \dots (6)$$

$$\underline{u}_0 = 0, \; \nabla \cdot \underline{u}_0 = 0 \qquad \dots (7)$$

$$V^2 V_0^i = -4\pi G\rho, \; \nabla^2 V_0^0 = 0. \qquad \dots (8)$$

These equations are simplified with $\partial/\partial z = 0$ and $\partial/\partial\phi = 0$ and the resulting system after simplification is solved. The solutions obtained are matched at $r = R_0$ across the boundary surface of the fluid. The finite solution can be easily found.

3. Perturbation State

Let the equilibrium state be perturbed, then every physical variable quantity can be expanded, as usual and has been used for the stability problems whether for cylindrical configurations or for other different models, in a series

$$Q(r, \phi, z; t) = Q_0(r) + \varepsilon Q_1(r, \phi, z). \qquad \dots (9)$$

Here the second term in the right side of (9) is the perturbed quantity and Q_1 stands for p, $\underline{u}$, V^i and V^0 and for the radial distance of the fluid cylinder. The amplitude ε of the perturbation is being

$$\varepsilon = \varepsilon_0 \exp(i\,\sigma\,t) \qquad \text{... (10)}$$

where ε_0 (= ε at t = 0) is the initial amplitude in the unperturbed state and σ is the growth rate. For a single Fourier term, the perturbed fluid interface is proposed to be

$$r = R_0 + \varepsilon_0 \exp[i\,(kz + m\phi + \sigma t)] \qquad \text{... (11)}$$

where k and m are the longitudinal and azimuthal wavenumbers respectively. The term $(r - R_0)$ in eqn. (11) is the elevation of the surface wave.

By the use of the expansions (9), the relevant perturbation equations coming out from the linearalization of the system (1)-(5) are given by

$$\partial \underline{u}_1/\partial t + (\underline{U} \cdot \underline{\nabla})\underline{u}_1 = - \underline{\nabla}\,\pi_1 + 2(\underline{u}_1 \wedge \underline{\Omega}) \qquad \text{... (12)}$$

$$\pi_1 = p_1/\rho - V_1^i \qquad \text{... (13)}$$

$$\underline{\nabla} \cdot \underline{u}_1 = 0 \qquad \text{... (14)}$$

$$\nabla^2 V_1^i = 0 \qquad \text{... (15)}$$

$$\nabla^2 V_1^0 = 0. \qquad \text{... (16)}$$

This system of equations is simplified on using the time dependence as given above by (10). From the view point of the linear theory and based on the linear perturbation technique, every perturbed quantity can be expressed as exp $[i\,(kz + m\phi + \sigma t)]$ times an amplitude function of r. Consequently, on solving (12), u_{1r}, $u_{1\phi}$ and u_{1z} are given as

$$u_{1r} = \frac{-(\sigma + ikU)}{[(\sigma + ikU)^2 + 4\Omega^2]}\frac{\partial \pi_1}{\partial r} + \frac{4im\,\Omega^2}{[(\sigma + ikU)^2 + 4\Omega^2]}\frac{\pi_1}{r} \qquad \text{... (17)}$$

$$u_{1\phi} = \frac{2}{[(\sigma + ikU)^2 + 4\Omega^2]}\frac{\partial \pi_1}{\partial \phi} + \frac{im\,\pi_1}{r(\sigma + ikU)}\left[\frac{4\Omega^2}{(\sigma + ikU)^2 + 4} - 1\right] \qquad \text{... (18)}$$

$$u_{1z} = -\frac{ik\,\pi_1}{(\sigma + ikU)}\,. \qquad \text{... (19)}$$

Combining eqns. (14), (17)-(19), we get

$$\frac{d^2\pi_1}{dr^2} + \frac{1}{r}\frac{d\pi_1}{dr} + \left(q - \frac{m^2}{r^2}\right) = 0 \qquad \text{... (20)}$$

$$q = k^2\left[1 + \frac{4\Omega^2}{(\sigma + ikU)^2}\right]. \qquad \text{... (21)}$$

The solution of eqn. (20) is given by

$$\pi_1 = A\,J_m\,(qr)\exp\,(i(kz + m\phi + \sigma t)) \qquad \text{... (22)}$$

where A is an arbitrary constant to be determined and J_m the ordinary Bessel function of first kind of order m.

Similarly eqns. (15) and (16), based on the linearized theory, are solved and first order perturbations V_1^i and V_1^0 are given by

$$V_1^i = B\,I_m\,(kr)\exp\,(i\,(kz + m\phi + \sigma t)) \qquad \text{... (23)}$$

$$V_1^0 = C\,K_m\,(kr)\exp\,(i(kz + m\phi + \sigma t)) \qquad \text{... (24)}$$

where I_m and K_m are the modified Bessel function of order m and B, C are constants of integrations to be determined.

4. Boundary Conditions and Dispersion Relation

The solutions given by eqns. (22)-(24) must satisfy the following boundary conditions :

(i) The normal component of the velocity of the fluid must be compatible with the velocity of the boundary surface at $r = R_0$.

(ii) The gravitational potential and its derivatives must be continuous across the surface.

(iii) The normal component of the total stress tensor must be continuous across the boundary surface from which we have the following dispersion relation :

$$\frac{[(\sigma + ikU) + 4\,\Omega^2]}{\left[yJ_m'\,(y) + \dfrac{2im}{(\sigma + ikU)}J_m\,(y)\right]}J_m\,(y) = 2\pi G\rho - \Omega^2 - 4\pi G\rho K_m\,(x)\,I_m\,(x) \qquad \text{... (25)}$$

where x is, the dimensional wavenumber, given by $x = kR_0$ and y is defined by $y = qR_0$.

5. Discussion and Conclusion

Equation (25) is the desired dispersion relation of gravitational streaming rotating fluid cylinder surrounded by self gravitating vacuum. It relates the growth rate σ with the streaming velocity U, angular velocity Ω, the wavenumbers x, y, m and other parameters ρ, G and R_0.

For non-rotating (i.e $\Omega = 0$) and non-streaming fluid (i.e $U = 0$) the dispersion relation (25) reduces to that of Chandrasekhar[4]. Moreover if we put $m = 0$, $\Omega = 0$, $U = 0$ in (25) we recover the relation derived for first time by Chandrasekhar and Fermi[1]. The latter authors derived their relations by using the principle of conservation of energy. A numerical and analytical study of the dispersion relation (25) show that the fluid cylinder is gravitationally stable for all non-axisymmetric perturbation $m \geq 1$. While in the axisymmetric mode $m = 0$ it is stable if

$$x \geq 1.0668 \qquad \text{... (26)}$$

and vice versa. The equality in (26) corresponds to the marginal stability.

In the absence of the streaming ($U = 0$), the dispersion relation (25) can be written in the dimensionless form

$$[ynJ_m'(y) + 2m\omega J_m(y)]\,[K_m(x)\,I_m(x) - 1/2 + \omega^2] + n(n^2 + 4\omega^2)\,J_m(y) = 0 \qquad \text{... (27)}$$

where the dimensionless quantity n, ω are defined as follow

$$n = \frac{\sigma}{(4\pi G\rho)^{1/2}}, \quad \omega = \frac{\Omega}{(4\pi G\rho)^{1/2}}. \qquad \text{... (28)}$$

If $x = 0$ the dispersion relation (27) takes the simpler form

$$n^2 - 2\omega n + m(\omega^2 - 1/2) + 1/2 = 0. \qquad \text{... (29)}$$

Hence

$$n = \omega \pm \sqrt{(m-1)(1/2 - \omega^2)}. \qquad \text{... (30)}$$

A neutral mode of oscillation is obtained if

$$\omega^2 = (m-1)/(2m). \qquad \text{... (31)}$$

It is clear that the angular velocity must satisfy the following

$$\sqrt{2}\,\omega \leq 1. \qquad \text{... (32)}$$

Neglecting the rotation effect the dispersion relation (25) reduces to

$$(\sigma + ikU)^2 = 4\pi G\rho\, x\, I_m'(x)/I_m(x)\,[I_m(x)\,K_m(x) - 1/2]. \qquad \text{... (33)}$$

The analytical discussions of this dispersion relation with the help of asymptotic behaviour of the modified Bessel's functions[7] show that the streaming has a destabilizing character. Its effect is to increase the unstable states in $m = 0$ mode and increasing the unstable regions for all modes. These results are confirmed through the numerical discussions of relation (33) as follows.

The unstable region is increasing with increasing U^* values. In fact not only the maximum mode of instability is increasing but also the critical value of the wavenumber. Corresponding to $U^* = 0, 0.1, 0.3, 0.5, 0.7, 0.9$, and 1.0 it is found that the maximum growth rate $\sigma^* = 0.2455, 0.3455, 0.5455, 0.7455, 0.9455, 1.146$ and 1.246 respectively.

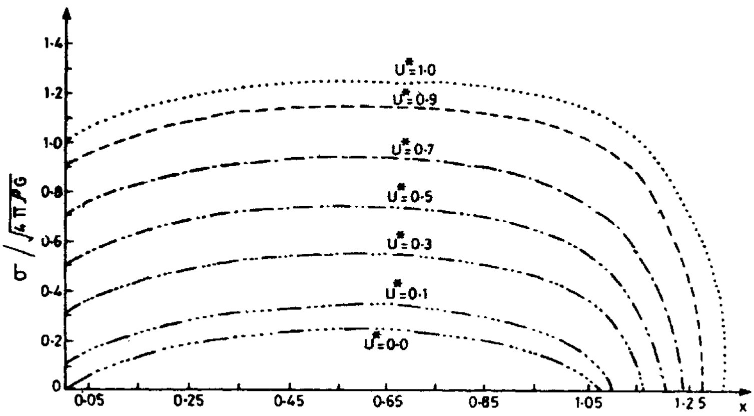

FIG. 1.

REFERENCES

1. S. Chandrasekhar and E. Fermi, *Astrophys. J.* **118** (1953), 116-41.
2. H. Robe, *Ann. Astrophys.* **31** (1969), 549.
3. H. Robe, *Astro. Astrophys.* **75** (1979) 14.
4. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford, 1981.
5. E. Pits and R. Tayler, *MNRAS* **216** (1985), 139.
6. A. Radwan and S. Elazab, *Pure and Appl. Sci.* **83** (1990), 154.
7. M. Abramowitz, and I. Stegun, *Handbook of Mathematical Functions* Dover Publ., New York, 1965.
8. S. Elazab, *J. Phys. Soc. Japan* **60** (1991), 1233.