Indian J. pure appl. Math., **25**(7) : 689-691, July 1994

A NOTE ON SOME *U*-NUMBERS*

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(Received 16 *April* 1993; *after revision* 3 *March* 1994;
accepted 25 *March* 1994)

In this short paper we will construct some U-numbers such that their L_m-sequences satisfy the given conditions.

First we recall the following.

Definition[2] — Let $\gamma \in C$ and $m \in Z^+$. The number γ is called an U_m-number if there are infinitely many polynomials $P_n(x) \in Z[x]$ ($\deg P_n(x) = m$) such that

$$0 < |P_n(\gamma)| \leq H(P_n)^{-w(n)} \qquad (n = 1, 2, ...), \quad \lim_{n \to \infty} w(n) = \infty \qquad ... (1)$$

and if there exist constants $c > 0$, $k > 0$ depending only on γ and m such that the inequality $|P(\gamma)| > cH(P)^{-k}$ hold for all polynomials $P(x) \in Z[x]$ of degree $< m$. (Here $H(P)$ denotes the maximum of the absolute values of the coefficients of $P(x)$). The sequence of polynomials $\{P_n(x)\}$ satisfying (1) is called a L_m-sequence for γ.

Theorem 1 — Let $g : Z^+ \to R^+$ be a function with $\lim_{n \to \infty} g(n) = \infty$ and $A = \{a_n\}$ be a sequence of positive integers with

$$0 < a_{n+1} - a_n < \frac{a_n}{g(n)} \qquad (n = 1, 2, ...). \qquad ... (2)$$

Next let $\{w(n)\}$ be a sequence of positive real numbers with $\lim_{n \to \infty} \sup w(n) = \infty$. Then there is a Liouville number ξ such that

$$\left| \xi - \frac{c_n}{d_n} \right| < d_n^{-w(n)} \qquad (n = 1, 2, ...) \qquad ... (3)$$

where $c_n, d_n \in A$.

* This paper is supported by the Scientific and Technical Research Counsil of Turkey.

PROOF : Let $a, b \in A$ with $a < b$. Since $\lim_{n \to \infty} g(n) = \infty$, there is an integer j_0 such that $g(n-1) > b^{w(1)}$ if $n > j_0$. Then we choose $a_m \in A$ with $aa_m > bj_0$ and $a_m^{w(2)} > 2b^{w(1)}$. Since $0 < a/b < 1$, there is an integer j with $j_0 < j \leq m$ such that $\frac{a_{j-1}}{a_m} < \frac{a}{b} \leq \frac{a_j}{a_m}$. ... (4)

Combining (2) and (4) we obtain $\left| \frac{a}{b} - \frac{a_{j-1}}{a_m} \right| < g(j-1)^{-1} < b^{-w(1)}$, or setting $a = c_1$, $b = d_1$, $a_{j-1} = c_2$, $a_m = d_2$ we have $\left| \frac{c_1}{d_1} - \frac{c_2}{d_2} \right| < d_1^{-w(1)}$. Now assume that rational numbers $\frac{c_i}{d_i} < 1$ $(i = 1, 2, ..., n)$ have been constructed. Let $j_1 > 0$ be an integer such that $g(k-1) > d_n^{w(n)}$ if $k > j_1$. Then we choose $d_{n+1} \in A$ with d_{n+1} $c_n, > d_n a_{j_1}$ $d_n < d_{n+1}$ and $d_{n+1}^{w(n+1)} > 2d_n^{w(n)}$. Using the above argument we find a integer $c_{n+1} \in A$ such that $c_{n+1} < d_{n+1}$ and $0 < \left| \frac{c_n}{d_n} - \frac{c_{n+1}}{d_{n+1}} \right| < d_n^{-w(n)}$. Thus we obtain a sequence $\left\{ \frac{c_n}{d_n} \right\}$ satisfying the relation

$$0 < \left| \frac{c_n}{d_n} - \frac{c_{n+1}}{d_{n+1}} \right| < d_n^{-w(n)} \qquad (n = 1, 2, ...).$$

Using this inequality and the conditions on d_n we see that

$$0 < \left| \frac{c_m}{d_m} - \frac{c_n}{d_n} \right| < 2\, d_n^{-w(n)} \qquad (m > n), \qquad ... (5)$$

that is $\left\{ \frac{c_n}{d_n} \right\}$ is a Cauchy sequence. Set $\xi = \lim_{n \to \infty} \frac{c_n}{d_n}$. Taking limit as $m \to \infty$ in (5) we get $0 < \left| \xi - \frac{c_n}{d_n} \right| \leq 2\, d_n^{-w(n)}$, and this completes the proof.

Remarks : (1) We note that Theorem 1 becomes false if we omit the condition $\lim_{n \to \infty} g(n) = \infty$ on g. To see this, let us take the sequence $A = \{2^n\}$. In this case $\{g(n)\}$ is bounded and $\xi \in Q$ if $\xi = \lim_{n \to \infty} \frac{c_n}{d_n}$ $(c_n, d_n \in A)$. Secondary let us consider the sequence $A = (2^2, 2^2+1, 2^2+2, 2^3, 2^3+1, 2^3+2, ..., 2^n, 2^n+1, 2^n+2, ...)$. We have for this sequence that $\lim_{n \to \infty} g(n) \neq \infty$, $\lim_{n \to \infty} \sup g(n) = \infty$ but $\xi \in Q$ if $\xi = \lim_{n \to \infty} \frac{c_n}{d_n}$.

(2) Let $\varepsilon > 0$ and $\frac{r}{s} \in Q$ with $0 < \frac{r}{s} < 1$. Using the argument in (4) we find

integers $a_i, a_t \in A$ such that $\left| \frac{r}{s} - \frac{a_i}{a_t} \right| < \varepsilon$. If we construct a Liouville numbers ξ with $c_1 = a_i$, $d_1 = a_t$ (choosing $w(1)$ with $d_1^{-w(1)} < \varepsilon$) we see that $\left| \xi - \frac{r}{s} \right| < 3\varepsilon$ and this shows that the set of all Liouville numbers in Theorem 1 is dense on [0, 1].

Theorem 2[1] — Let p_n denote the nth prime. Then there is a real number α with $0 < \alpha < 1$ such that

$$p_{n+1} - p_n < p_n^{\alpha}.$$

It is clear that we can take $A = \{p_n\}$ in Theorem 1 and so we obtain.

Corollary 1 — There is a subset K of the set of all Liouville numbers which is dense in [0, 1] such that if $\xi \in K$ then

$$\left| \xi - \frac{a_n}{b_n} \right| < b_n^{-w(n)}, \quad \lim_{n \to \infty} w(n) = \infty,$$

where a_n, b_n are prime.

Now we take $A = \{p_n\}$ and choose the sequence $\{w(n)\}$ with $w(n+1) > w(n) > n$ in Theorem 1. Set $\gamma = \xi^{1/m}$ where $\xi \in K$. If we define the polynomials $P_n(x)$ as $P_n(x) = b_n x^m - a_m$ $(m \in Z^+)$ we obtain that $0 < | P_n(\gamma) | < 2H(P_n)^{-w(n)} < H(P_{n+1})^{-\rho}$, where $\rho = 1 - \alpha - \varepsilon > 0$ for some $\varepsilon > 0$. Thus γ satisfies the conditions in Theorem 5 in LeVeque[2] and so we have $\gamma \in U_m$. It is clear that $\{P_n(x)\}$ is a L_m-sequence for γ and non-zero coefficients of $P_n(x)$ are prime.

Finally let $A = \{a_n\}$, where $a_n = f(n)$, $f(x) \in Z[x]$ (deg $f = k \geq 1$) and $\varepsilon < 1$. Then for large n we have $0 < | f(n+1) - f(n) | < | f(n) |^{(k-1+\varepsilon)/k}$. Hence applying Theorem 1 with $a_n = f(n)$ we obtain :

Corollary 2 — Let $f(x) \in Z[x]$ with deg $f \geq 1$. Then there are infinitely many Liouville numbers ξ with

$$\left| \xi - \frac{a_n}{b_n} \right| < | b_n |^{-w(n)} \text{ where } a_n, b_n \in \{f(n)\} \text{ and } \lim_{n \to \infty} w(n) = \infty.$$

ACKNOWLEDGEMENT

The author would like to thank the referee for his valuable criticism.

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