Indian J. pure appl. Math., 25(7) : 693-697, July 1994

A 2-BASE OF THE VARIETY OF QUASI-COMMUTATIVE BCI-ALGEBRAS

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(*Received* 31 *December* 1992; *after revision* 20 *October* 1993; *accepted* 24 *March* 1994)

In this note, we give a 2-base of the variety of quasi-commutative BCI-algebras of type $(m, n; k, l)$. The 2-base is
(I) $u * (((x * y) * (x * z)) * (z * y)) = u$,
(II) $((x *^{1+m} (x * y)) *^n (y * x) = ((y *^{1+k} (y * x)) *^l (x * y)) * 0$.
As consequences we obtain a 2-base for the variety of implicative BCI-algebras and I-variety, respectively.

Yutani[14] introduced the concept of quasi-commutativity for BCK-algebras and proved that commutative BCK-algebras, positive implicative BCK-algebras, implicative BCK-algebras and finite BCK-algebras were quasi-commutative. In particular, he also showed that quasi-commutative BCK-algebras form a variety. Iséki[5] generalized the concept to BCI-algebras. Lei and Xi[7] pointed out that p-semisimple BCI-algebras are quasi-commutative of types (0, 1; 0, 0) and (0, 2; 1, 0), but need not be of (1, 0; 0, 0). Hoo[2, 3] studied further properties of such algebras. Meng and Xin[12] introduced implicative BCI-algebras and proved that such algebras are quasi-commutative of type (0, 0; 0, 1). Iséki[6] introduced a variety of BCI-algebras (called I-variety[9]), each algebra of it is quasi-commutative of type (0, 0; 1, 0).

For an algebraic variety it is important to look for minimal equational bases, in the setting of BCK-algebras the question have been discussed by researchers, see Cornish[1], Palasinski and Wozniakowska[13], Idziak[4], and Meng[8]. But for BCI-algebras, few people study this question. Meng and Xin[11] gave a 3-base for the variety of commutative BCI-algebras. Recently, Meng[9] proved that this variety is 2-based. In a earlier paper[10], we obtain a 2-base of the variety of medial BCI-algebras.

In view of the above facts one may ask if the variety of quasi-commutative BCI-algebms has a 2-base. This note give a positive solution.

Now let us recall some definitions and results, which are needed for the development of this paper.

By a BCI-algebra we mean a set X with a binary operation $*$ and a nullary operation 0 satisfying the following conditions :

BCI-1 $((x * y) * (x * z)) * (z * y) = 0$,

BCI-2 $(x * (x * y)) * y = 0$,

BCI-3 $x * x = 0$,

BCI-4 $x * y = 0$ and $y * x = 0$ imply $x = y$.

In what follows, X would always mean a BCI-algebra and m, n, k, l would mean non-negative integers unless otherwise specified.

For any non-negative integer n, $x *^n y$ is recursively defined as follows :

(i) $x *^0 y = x, \quad x *^1 y = x * y$,

(ii) $x *^{n+1} y = (x *^n y) * y$.

Lemma 1 — For any algebra $(X; *, 0)$ of type (2, 0) if $0 = 0 * (0 * 0)$ then for all $m \geq 0$,

$$0 *^{1+m} (0 * 0) = 0. \qquad \text{... (1)}$$

PROOF : We proceed by induction on m. If $m = 0$, then by hypothesis we know that (1) holds. Now suppose that the assertion is true for k, then $0 *^{1+k+1} (0 * 0)$ $(0 *^{1+k} (0 * 0)) * (0 * 0) = 0 * (0 * 0) = 0$. This says that (1) holds for $k + 1$. The proof is complete.

As an immediate consequence of Lemma 1, we have

Lemma 2 — For any algebra $(X; *, 0)$ of type (2, 0), if $0 = 0 * (0 * 0)$, then for all $m \geq 0$ and $n \geq 0$,

$$0 *^{1+m} (0 * 0) *^n (0 * 0) = 0 \qquad \text{... (2)}$$

Definition 1[14] — A BCI-algebra $(X; *, 0)$ is said to be quasi-commutative of type $(m, n; k, l)$ if it satisfies

(Q) $(x *^{1+m} (x * y)) *^n (y * x) = (y *^{1+k} (y * x)) *^l (x * y)$

for any x, y in X.

The main result of this paper is the following.

Theorem 3 — An algebra $(X; *, 0)$ of type (2, 0) is a quasi-commutative BCI-algebra of type $(m, n; k, l)$ if and only if it satisfies

(I) $u * (((x * y) * (x * z)) * (z * y)) = u$,

(II) $(x *^{1+m} (x * y)) *^n (y * x) = ((y *^{1+k} (y * x)) *^l (x * y)) * 0$,

for any x, y, z and u in X.

PROOF : The direction ($\Rightarrow$) is obvious.

($\Leftarrow$) For convenience of notation, denote $\theta = ((0 * 0) * (0 * 0)) * (0 * 0)$. Let $x = y = z = 0$ and $u = \theta$ in (I), then

$$\theta = \theta * \theta. \qquad \text{... (3)}$$

Putting $x = y = z = \theta$ in (I), and using (3), we have

$$u = u * (((\theta * \theta) * (\theta * \theta)) * (\theta * \theta))$$

$$= u * ((\theta * \theta) * \theta)$$

$$= u * (\theta * \theta)$$

$$= u * \theta,$$

that is,

$$u = u * \theta. \quad \text{... (4)}$$

Assuming $u = 0$ and $u = 0 * 0$ in (4) respectively, we obtain

$$0 = 0 * \theta, \quad \text{... (5)}$$

$$0 * 0 = (0 * 0) * \theta. \quad \text{... (6)}$$

In (I), substituting $y = z = \theta$ and then using (3) and (5), we have

$$u = u * (((0 * \theta) * (0 * \theta)) * (\theta * \theta))$$

$$= u * ((0 * 0) * \theta)$$

$$= u * (0 * 0),$$

hence

$$u = u * (0 * 0). \quad \text{... (7)}$$

When $u = 0$ we obtain

$$0 = 0 * (0 * 0). \quad \text{... (8)}$$

If we let $x = y = 0$ in (II) and use (8) and Lemma 2, we obtain $0 = 0 * 0$, hence by (7)

$$u = u * 0. \quad \text{... (9)}$$

(Q) follows from (9) and (II).

Assuming $y = z = 0$ in (I) and using (9) we have

$$u = u * (((x * 0) * (x * 0)) * (0 * 0))$$

$$= u * ((x * x) * 0)$$

$$= u * (x * x),$$

thus

$$u = u * (x * x). \quad \text{... (10)}$$

In this identity letting $u = x * x$, then we have

$$x * x = (x * x) * (x * x) = (x * x) * ((x * x) * 0). \quad \text{... (11)}$$

Multiplying both sides of (11) on the right hand side by $(x * x) * 0$, we obtain

$$(x * x) * ((x * x) * 0) = (x * x) *^2 ((x * x) * 0),$$

or

$$x * x = (x * x) *^2 ((x * x) * 0).$$

Repeating the above argument m times we obtain

$$x * x = (x * x) *^{1+m} ((x * x) * 0).$$

In the similar way, we have

$$x * x = ((x * x) *^{1+m} ((x * x) * 0)) *^n (0 * (x * x)).$$

Combining (Q) we obtain

$$x * x = (0 *^{1+k} (0 * (x * x))) *^l ((x * x) * 0).$$

From (10), we have $0 * (x * x) = 0$ and from (11), we have $(x * x) * 0 = x * x$, hence

$$x * x = 0 *^l (x * x) = 0,$$

BCI-3 follows. In (I), letting $u = ((x * y) * (x * z)) * (z * y)$ and using BCI-3, we obtain

$$\begin{aligned}&((x * y) * (x * z)) * (z * y)\\ &= (((x * y) * (x * z)) * (z * y)) * (((x * y) * (x * z)) * (z * y))\\ &= 0,\end{aligned}$$

that is, BCI-1 holds. Putting $y = 0$ in BCI-1 and using (9), we obtain BCI-2. If $x * y = 0$ and $y * x = 0$ then by (9) and (Q) we have

$$\begin{aligned}x &= (x *^{1+m} 0) *^n 0\\ &= (x *^{1+m} (x * y)) *^n (y * x)\\ &= (y *^{1+k} (y * x)) *^l (x * y)\\ &= (y *^{1+k} 0) *^l 0\\ &= y,\end{aligned}$$

this says that BCI-4 is true. Putting the above results together we have proved that $(X; *, 0)$ is a quasi-commutative BCI-algebra of type $(m, n; k, l)$. This completes the proof.

This theorem shows that the variety of quasi-commutative BCI-algebras of type $(m, n; k, l)$ is 2-based and an equational base is given by (I) and (II).

Definition 2[12] — A BCI-algebra $(X; *, 0)$ is called implicative if it satisfies for all x, y in X,

$$x * (x * y) = (y * (y * x)) * (x * y).$$

Implicative BCI-algebras form a variety (Meng and Xin[12], Theorem 7). If we assume $m = n = k = 0$ and $l = 1$, by Theorem 3 we have

Theorem 4 — The variety of implicative BCI-algebras is 2-based and an equational base is given by (I) and (III)

(III) $x * (x * y) = ((y * (y * x)) * (x * y)) * 0.$

Definition 3[6] — By a I-variety we mean an algebraic variety of type (2, 0) satisfying the following

BCI-1 $((x * y) * (x * z)) * (z * y) = 0,$

BCI-2 $(x * (x * y)) * y = 0,$

BCI-3 $x * x = 0,$

(9) $x = x * 0,$

(12) $(x * y) * z = (x * z) * y,$

(13) $x * (x * y) = (y * (y * x)) * (y * x).$

If $m = n = l = 0$ and $k = 1$ then by Theorem 3 we obtain that (I) and

(IV) $x * (x * y) = ((y * (y * x)) * (y * x)) * 0$

provide two equations for the base of the I-variety. Hence an I-variety is 2-based.

ACKNOWLEDGEMENT

The authors are grateful to the referee for useful comments.

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