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PROPERTIES OF THE OSCILLATING AND NONOSCILLATING SOLUTIONS OF OPERATOR-DIFFERENTIAL EQUATIONS

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In the present paper the properties of the oscillating solutions of a class of operator-differential equations are investigated. Effective sufficient conditions for nonoscillation of all solutions of the operator-differential equations considered are found.

1. INTRODUCTION

In the present paper the oscillatory and asymptotic properties of the solutions of a class of operator-differential equations are investigated. A single approach is introduced to the investigation of the oscillatory and asymptotic properties of the solutions of ordinary differential equations with "maximum", with distributed delay, with autoregulable deviation, etc. Sufficient, conditions for nonoscillation of all solutions are obtained, as well as conditions under which the oscillating solutions of the equation considered enjoy certain asymptotic properties.

We shall note that the results obtained generalize some results obtained in Grace and Lalli[2].

2. PRELIMINARY NOTES

Consider the operator-differential equation

$$[\tau_{n-1}(t)\,[\tau_{n-2}(t)\,[\dots[\tau_1(t)\,x'(t)]'\dots]'\,]'\,]' + F(t_1\, x(t), (\mathcal{A}x)(t)) = C(t) \qquad \dots (1)$$

where $\mathcal{A}$ is an operator with certain properties.

Let $t_0 \in \mathbb{R}$ be a fixed number and $\tau_i \in C([t_0, \infty), (0, \infty))$, $(1 \le i \le n-1)$.

Introduce the following notation :

$$(L_0x)(t) = x(t)$$

$$(L_ix)(t) = \tau_i(t)\,[\,(L_{i-1}x)(t)]', \qquad (1 \le i \le n), \quad \tau_n(t) := 1.$$

Denote by $\mathcal{D}_n$ the set of all functions $x \in C([T_x, \infty); \mathbb{R})$, $T_x \ge t_0$ such that the functions $L_ix\ (1 \le i \le n)$ exist and are continuous in $[T_x, \infty)$.

Definition 1 — The function $x : [T_x, \infty) \to \mathbb{R}$ is said to be a solution of eqn. (1) if $x \in \mathcal{D}_n$ and x satisfies eqn. (1) for $t \ge \max\{T_x, T_{Ax}\}$.

Definition 2 — A given function $u : [t_0, \infty) \to \mathbb{R}$ is said to eventually enjoy the property P if there exists a point $t_{P,u} \ge t_0$ such that for $t \ge t_{P,u}$ it enjoys the property P.

Definition 3 — The solution x of eqn. (1) is said to be regular if $\sup |x(t)| > 0$, eventually.

Definition 4 — The regular solution x of eqn. (1) is said to oscillate if $\sup\{t,\ x(t) = 0\} = \infty$. Otherwise the regular solution x is said to be nonoscillating.

We shall say that conditions (H) hold if the following conditions are met :

H1. $\tau_i \in C([t_0, \infty); (0, \infty))$, $(1 \le i \le n-1)$.

H2. $C \in C([t_0, \infty); \mathbb{R})$.

H3. $F \in C([t_0, \infty) \times \mathbb{R}^2; \mathbb{R})$.

H4. There exists a function $G(t, u, v)$ such that

$$G(t, u, v) \in C([t_0, \infty) \times (0, \infty)^2; (0, \infty))$$

$$|F(t, u, v)| \le G(t, |u|, |v|); \quad G(t, u_1, v_1) \le G(t, u_2, v_2)$$

for $\qquad u_1 \le u_2, \quad v_1 \le v_2.$

H5. $\mathcal{A} : \mathcal{D}_n \to C([T_{AX}, \infty); \mathbb{R})$, $T_{AX} \ge t_0$.

H6. For any two functions $\varphi, \psi \in \mathcal{D}_n$ such that $\varphi(t) \le \psi(t)$ eventually the inequality $(\mathcal{A}\varphi)(t) \le (\mathcal{A}\psi)(t)$ is eventually valid.

H7. For any function $\varphi \in C([t_0, \infty); \mathbb{R})$ for which $\varphi(t) > 0 < \varphi(t) < 0)$ eventually, the inequality $(\mathcal{A}\varphi)(t) \ge 0 < (\mathcal{A}\varphi)(t) \le 0)$ is eventually valid.

3. Main Results

Theorem 1 — Let the following conditions hold :

1. Conditions (H) are satisfied.

2. $$\int^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \dots \int_{s_{n-2}}^{\infty} \frac{1}{\tau_{n-1}(s_{n-1})}$$

$$\times \int_{s_{n-1}}^{\infty} G(s, cm(s), (\mathcal{A}\, cm)(s))\, ds\, ds_{m-1} \dots ds_1 < \infty$$

c = const. > 0.

3. $$\int^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \dots \int_{s_{n-2}}^{\infty} \frac{1}{\tau_{n-1}(s_{n-1})}$$

$$\times \int_{s_{n-1}}^{\infty} |C(s)|\, ds\, ds_{n-1} \dots ds_2\, ds_1 < \infty.$$

Then each oscillating solution x of eqn. (1) such that

$$|x(t)| = O(m(t))_{t \to \infty} \qquad \dots (2)$$

where $m \in C([t_0, \infty), (0, \infty))$, enjoys the property

$$\lim_{t \to \infty} (L_i x)(t) = 0. \qquad \dots (3)$$

PROOF : Let x be an oscillating solution of eqn. (1) for which condition (2) is met. From (2) it follows that there exists a constant $c > 0$ such that

$$0 < |x(t)| \le cm(t).$$

From conditions H6 and H7 we obtain that

$$|(\mathcal{A} x)(t)| \le (\mathcal{A}\,|x|)(t) \le (\mathcal{A}\, cm)(t).$$

From the fact that x is an oscillating solution of eqn. (1) for $t \ge T$ $(T \ge t_0)$ it follows that $L_i x$ $(1 \le i \le n-1)$ are oscillating functions.

Let $\{t_k\}_{k=1}^{\infty}$ be a sequence of numbers such that $(L_{n-1} x)(t_k) = 0$. Let $\alpha_k \in (t_k, t_{k+1})$ and $|(L_{n-1} x)(\alpha_k)| = \max_{t_k < t \le t_{k+1}} |(L_{n-1} x)(t)|$.

Integrate (1) from t_k to α_k and obtain that,

$$(L_{n-1} x)(\alpha_k) - (L_{n-1} x)(t_k) = -\int_{t_k}^{\alpha_k} F(s, x(s), (\mathcal{A}x)(s))\, ds + \int_{t_k}^{\alpha_k} C(s)\, ds$$

$$|(L_{n-1} x)(\alpha_k)| \le \int_{t_k}^{\alpha_k} |F(s, x(s), (\mathcal{A}x)(s))|\, ds + \int_{t_k}^{\alpha_k} |C(s)|\, ds$$

$$\le \int_{t_k}^{\alpha_k} G(s, |x(s)|, |(\mathcal{A}x)(s)|)\, ds + \int_{t_k}^{\alpha_k} |C(s)|\, ds$$

$$\leq \int_{t_k}^{\alpha_k} G(s, cm(s), (\mathcal{A}\, cm)(s))\, ds + \int_{t_k}^{\alpha_k} |\, C(s)\, |\, ds.$$

A summation with respect to k yields

$$\sum_{k=1}^{\infty} |\, (L_{n-1}x)(\alpha_k)\, | \leq \int_{t_1}^{\infty} G(s, cm(s), (\mathcal{A}\, cm)(s))\, ds$$

$$+ \int_{t_1}^{\infty} |\, C(s)\, |\, dx < \infty.$$

From $\lim_{k\to\infty} (L_{n-1}x)(d_k) = 0$ it follows that $\lim_{t\to\infty} (L_{n-1}x)(t) = 0$.

Integrate (1) from t to ∞ and obtain that

$$(L_{n-1}x)(t) = \int_{t}^{\infty} F(s, x(s), (\mathcal{A}x)(s))\, ds - \int_{t}^{\infty} C(s)\, ds. \qquad \text{... (4)}$$

We shall prove that $\lim_{t\to\infty} (L_{n-2}x)(t) = 0$. Let $\{t_k'\}_{k=1}^{\infty}$ be a sequence of numbers such that $(L_{n-2}x)(t_k') = 0$ and let $\alpha_k' \in (t_k', t'_{k+1})$ so that

$$|\, (L_{n-2}x)(\alpha_k')\, | = \max_{t'_k \leq t \leq t'_{k+1}} |\, (L_{n-2}x)(t)\, |.$$

Integrate (4) from t_k' to α_k' and obtain

$$(L_{n-2}x)(\alpha_k') = -\int_{t_k'}^{\alpha_k'} \frac{1}{\tau_{n-1}(t)} \int_{t}^{\infty} F(s, x(s), (\mathcal{A}x)(s))\, ds\, dt$$

$$-\int_{t_k'}^{\alpha_k'} \frac{1}{\tau_{n-1}(t)} \int_{t}^{\infty} C(s)\, ds\, dt$$

$$|\, (L_{n-2}x)(\alpha_k')\, | \leq \int_{t_k'}^{\alpha_k'} \frac{1}{\tau_{n-1}(t)} \int_{t}^{\infty} G(s, cm(s), (\mathcal{A}cm)(s))\, ds\, dt$$

$$+ \int_{t_k'}^{\alpha_k'} \frac{1}{\tau_{n-1}(t)} \int_{t}^{\infty} |\, C(s)\, |\, ds\, dt.$$

We sum up with respect to k and obtain

$$\sum_{k=1}^{\infty} \mid (L_{n-2}x)(\alpha_k') \mid \le \int_{t_1^1}^{\infty} \frac{1}{\tau_{n-1}(t)} \int_t^{\infty} G(s, cm(s), (\mathcal{A}cm)(s))\, ds\, dt$$

$$+ \int_{t_k^1}^{\infty} \frac{1}{\tau_{n-1}(t)} \int_t^{\infty} \mid C(s) \mid ds\, dt < \infty$$

i.e. $\lim_{t \to \infty} (L_{n-2}x)(t) = 0.$

Integrate (4) from t to ∞

$$(L_{n-2}x)(t) = - \int_t^{\infty} \frac{1}{\tau_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} F(s, x(s), (\mathcal{A}x)(s))\, ds\, ds_{n-1}$$

$$+ \int_t^{\infty} \frac{1}{\tau_{n-1}(s_{n-1})} \int_{s_{n-1}}^{\infty} C(s)\, ds\, ds_{n-1}.$$

By means of analogous arguments we conclude that

$$\lim_{t \to \infty} (L_i x)(t) = 0, \quad (0 \le i \le n-1) \ \square$$

Corollary 1 — Let the following conditions hold :

1. $\int^{\infty} \frac{dt}{\tau_i(t)} < \infty$, $1 \le i \le n-1$.

2. $\int^{\infty} G(s, cm(s), (\mathcal{A}cm)(s))\, ds < \infty$, c = const. > 0.

3. $\int^{\infty} |C(s)|\, ds < \infty$.

4. Conditions (H) are met.

Then each oscillating solution x of eqn. (1) which satisfies (2) enjoys property (3).

PROOF : Conditions 1 and 2 of Corollary 1 imply condition 1 of Theorem 1. From conditions 1 and 3 of Corollary 1 there follows condition 2 of Theorem 1. $\square$

Corollary 2 — Let the following conditions hold :

1. Conditions 1 and 2 of Corollary 1 are satisfied.

2. The function $\dfrac{C(t)}{G(t, cm(t), (\mathcal{A}\, cm)(t)}$ is bounded.

3. $G(t, cm(t), (\mathcal{A}\ cm)\ (t)) > 0$ for $t \ge t_0$.

4. Conditions (H) hold.

Then each oscillating solution x of eqn. (1) which satisfies (2) enjoys property (3).

PROOF : We shall prove that from conditions 2 and 3 of Corollary 2 there follows condition 3 of Corollary 1. From condition 2 of Corollary 2 it follows that there exists a constant $\mathcal{M} > 0$ such that

$$\frac{| C(t) |}{G(t, cm(t), (\mathcal{A}\, cm)(t))} \le \mathcal{M}.$$

Then

$$\int^{\infty} | C(t) | \, dt \le \mathcal{M} \int^{\infty} G(t, cm(t), (\mathcal{A}\, cm)(t))\, dt < \infty$$

which is just condition 3 of Corollary 1. □

Theorem 2 — Let the following conditions hold :

1. Conditions (H) are met.
2. Conditions 2 and 3 of Theorem 1 are satisfied.
3. There exists a constant $k > 0$ and a number $T \ge t_0$ such that

$$\lim_{t \to \infty} \inf \int_T^t [C(s) - G(s, k, (\mathcal{A}k)(s))]\, ds > 0$$

or

$$\lim_{t \to \infty} \sup \int_T^t [C(s) - G(s, k, (\mathcal{A}k)(s))]\, ds < 0.$$

Then all solutions x of eqn. (1) which satisfy (2) are non-oscillating.

PROOF : Let x be an oscillating solution of eqn. (1) which satisfies (2). From Theorem 1 it follows that $\lim_{t \to \infty} (L_k x)(t) = 0$, $(0 \le k \le n-1)$.

This fact allows us to conclude that there exists a number $T \ge t_0$ such that $(L_{n-1} x)(T) = 0$ and $|x(t)| \le k$ for $t \ge T$.

From conditions H6 and H7 we obtain that

$$|(\mathcal{A}\, x)(t)| \le (\mathcal{A}\, |x|)(t) \le (\mathcal{A}\, k)(t).$$

Then

$$| F(t, x(t), (\mathcal{A}\, x)(t)) | \le G(t, k, (\mathcal{A}\, k)(t)) |. \quad \text{... (5)}$$

From eqn. (1) and inequality (5) it follows that

$$C(t) - G(t, k, (\mathcal{A}\, k)(t)) \le (L_{n-1} x)(t) \le C(t) + G(t, k, (\mathcal{A}\, k)(t)). \quad \text{... (6)}$$

Integrate (6) from T to t and obtain

$$\int_T^t [C(s) - G(s, k, (\mathcal{A} k)(s))]\, ds \le (L_{n-1} x)(t)$$

$$\le \int_T^t [C(s) + G(s, k, (\mathcal{A} k)(s))]\, ds. \quad \ldots (7)$$

From condition 3 of Theorem 2 and inequalities (7) we obtain a contradiction with the assumption that x is an oscillating solution of eqn. (1). □

Theorem 3 — Let the following conditions hold :

1. Conditions (H) are satisfied.
2. There exists a function $H \in C([t_0, \infty); \mathbb{R})$ such that

$$\limsup_{\substack{v \to \infty \\ v \le u}} \frac{G(t, u, \mathcal{A} u)}{v} \le H(t).$$

3. $\int^\infty \frac{1}{\tau_1(s_1)} \int_{s_1}^\infty \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-2}}^\infty \frac{1}{\tau_{n-1}(s_{n-1})} \int_{s_{n-1}}^\infty H(s)\, ds\, ds_{n-1} \ldots ds_2\, ds_1 < \infty.$

4. Conditions 2 and 3 of Theorem 1 for $m(t) = k > 0$ hold.

Then each oscillating solution of eqn. (1) enjoys property (3).

PROOF : We shall prove that all oscillating solutions of eqn. (1) are bounded.

Suppose that x is an oscillating solution of eqn. (1) such that $\limsup_{t \to \infty} |x(t)| = \infty$. Then there exists a sequence of intervals $\{(\alpha_m, \beta_m)\}_{m=1}^\infty$ such that

$$\lim_{m \to \infty} \alpha_m = \lim_{m \to \infty} \beta_m = \infty, \ x(\alpha_m) = x(\beta_m), \ |x(t)| > 0 \text{ for } t \in (\alpha_m, \beta_m).$$

Introduce the notation

$$\mathcal{M}_m = \max_{\alpha_m \le t \le \beta_m} |x(t)| = |x(t_m)|, \ t_m \in (\alpha_m, \beta_m).$$

From the assumption that x is an unbounded function it follows that $\mathcal{M}_m \ge k$ and $\lim_{m \to \infty} \mathcal{M}_m = \infty$.

Let $P_1 < P_2 < \ldots < P_{n-1}$ be respectively the zeros of $(L_1 x)(t)$, $(L_2 x)(t)$, ..., $(L_{n-1} x)(t)$ and $\beta_m < P_{n-1}$.

Introduce the notation $\mathcal{N}_m = \max_{t \le P_{n-1}} |x(t)|$.

Integrate n times eqn. (1) and obtain

$$\mathcal{M}_m = |x(t_m)| \le \int_{\alpha_m}^{\beta_m} \frac{1}{\tau_1(s_1)} \int_{s_1}^{p_1} \frac{1}{\tau_2(s_2)} \cdots$$

$$\int_{s_{n-1}}^{p_{n-1}} G(s, \mathcal{N}_m, (\mathcal{A}\mathcal{N}_m)(s))\, ds \ldots ds_2\, ds_1$$

$$+ \int_{\alpha_m}^{\beta_m} \frac{1}{\tau_1(s_1)} \int_{s_1}^{p_1} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{p_{n-1}} |C(s)|\, ds \ldots ds_2\, ds_1$$

$$\le \int_{\alpha_m}^{\beta_m} \frac{1}{\tau_1(s_1)} \int_{s_1}^{p_{n-1}} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{p_{n-1}} G(s, \mathcal{N}_m, (\mathcal{A}\mathcal{N}_m)(s))\, ds \ldots ds_2\, ds_1$$

$$+ \int_{\alpha_m}^{\beta_m} \frac{1}{\tau_1(s_1)} \int_{s_1}^{p_{n-1}} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{p_{n-1}} |C(s)|\, ds \ldots ds_2\, ds_1$$

$$\le \int_{\alpha_m}^{\beta_m} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} G(s, \mathcal{N}_m, (\mathcal{A}\mathcal{N}_m)(s))\, ds \ldots ds_2\, ds_1$$

$$+ \int_{\alpha_m}^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} |C(s)|\, ds \ldots ds_2\, ds_1.$$

From condition 2 of Theorem 3 it follows that

$$1 \le \int_{\alpha_m}^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} \frac{G(s, \mathcal{N}_m, (\mathcal{A}\mathcal{N}_m)(s))}{\mathcal{M}_m}\, ds \ldots ds_2\, ds_1$$

$$+ \frac{1}{\mathcal{M}_m} \int_{\alpha_m}^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} |C(s)|\, ds \ldots ds_2\, ds_1$$

$$1 \le \int_{\alpha_m}^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} |H(s)|\, ds \ldots ds_2\, ds_1$$

$$+ \frac{1}{\mathcal{M}_m} \int_{\alpha_m}^{\infty} \frac{1}{\tau_1(s_1)} \int_{s_1}^{\infty} \frac{1}{\tau_2(s_2)} \cdots \int_{s_{n-1}}^{\infty} |C(s)|\, ds \ldots ds_2\, ds_1 \underset{m \to \infty}{\longrightarrow} 0$$

i.e. we obtain a contradiction.

From the contradiction obtained it follows that x is a bounded oscillating solution of eqn. (1) which satisfies (2) for $m(t) = k > 0$. Then from Theorem 1 there follows the assertion of Theorem 3. □

Theorem 4 — Let the following conditions hold :

1. Conditions (H) are met.
2. Conditions 2, 3 and 4 of Theorem 3 are fulfilled.
3. Conditions 3 of Theorem 2 is met.

Then all solutions x of eqn. (1) are nonoscillating.

Let x be an oscillating solution of equation (1). From Theorem 3 it follows that $\lim_{t \to 0} (L_k x)(t) = 0 < 0 \le k \le n-1)$. Further on the proof of Theorem 4 is carried out analogously to the proof of Theorem 2.

4. Some Particular Realizations of the Operator

Theorem 5 — Let the following conditions hold :

1. $(\mathcal{A}x)(t) = \max_{P(t) \le s \le q(t)} g(d(s) \cdot x(h(s)))$

1.1. $h, p, q \in C([t_0, \infty); \mathbb{R})$, $\lim_{t \to \infty} P(t) = \lim_{t \to \infty} h(t) = \infty$

1.2. $P(t) \le q(t)$ for $t \ge t_0$, $d \in C([t_0, \infty); (0, \infty)]$

1.3. $g \in C(\mathbb{R}; \mathbb{R})$, sgn $g(u)$ = sgn u, $g(u)$ is a nondecreasing function in $\mathbb{R}$.

2. Conditions H1-H4 hold.
3. Conditions 2 and 3 of Theorem 1 are met.

Then each oscillating solution x of the equation

$$(L_n x)(t) + F(t, x(t), \max_{P(t) \le s \le q(t)} g(d(s) \cdot x(h(s)))) = C(t) \qquad \ldots (8)$$

which satisfies (2) enjoys property (3).

Theorem 5 follows from Theorem 1 since it is immediately verified that condition 1 of Theorem 5 implies conditions H6 and H7. The validity of condition H5 is proved analogously to Lemma 1 of Angelov and Bainov[1].

Example 1 — Consider the differential equation

$$[e^t [e^{-t} [e^t \cdot x'(t)]']']' + e^{-t} \cdot x^{\alpha}(g(t))$$
$$= -6e^{-t} \cos t - 8 e^{-t} \sin t + e^{-t - 2\alpha g(t)} \cdot \sin^{\alpha} g(t)$$

where $\alpha > 0$, $g(t) > 0$ for $t \ge t_0 > 0$ and $\lim_{t \to \infty} g(t) = \infty$, $g \in C([t_0, \infty); (0, \infty)]$.

The functions $\tau_1(t) = \tau_3(t) = e^t$, $\tau_2(t) = e^{-t}$, $F(t, u, v) = e^{-t} \cdot v^{\alpha}$

$(\mathcal{A}x)(t) = x^{\alpha}(g(t))$, $C(t) = -6 e^{-t} \cos t - 8 e^{-t} \sin t + e^{-t - 2\alpha g(t)} \sin^{\alpha} g(t)$ satisfy the conditions of Theorem 5 for $m(t) = k$ = const > 0.

Then each oscillating solution x for which $|x(t)| \le c \cdot k$ eventually enjoys property (3). For instance, $x(t) = \frac{\sin t}{e^{2t}}$ is an oscillating solution enjoying property (3).

Theorem 6 — Let the following conditions hold :

1. $(\mathcal{A}x)(t) = \int_{P(t)}^{q(t)} K(t, s, x(t), x(s))\, d_s \tau(t, s)$

1.1. $p, q \in C([t_0, \infty); \mathbb{R})$, $\lim_{t \to \infty} P(t) = \infty$, $p(t) \leq q(t)$

1.2. $k \in C([t_0, \infty)^2 \times \mathbb{R}^2; \mathbb{R})$, the function $k(t, s, u, v)$ is nondecreasing with respect to u and to v.

1.3. $\operatorname{sgn} K(t, s, u, 0) = \operatorname{sgn} u$; $\operatorname{sgn} K(t, s, 0, v) = \operatorname{sgn} v$.

1.4. For each $t \geq t_0$ the function $\tau(t, s)$ is increasing with respect to 5.

1.5. The functions $\tau(t, p(t))$ and $\tau(t, q(t))$ are continuous and for $t' \geq t_0$ the following relation is valid

$$\lim_{t \to t'} \int_{\max\{P(t), P(t')\}}^{\min\{q(t), q(t')\}} |\tau(t', s) - \tau(t, s)|\, ds = 0$$

2. Conditions H1-H4 are met.

3. Condition 3 of Theorem 5 is satisfied.

Then each oscillating solution x of the equation

$$(L_n x)(t) + F(t, x(t), \int_{p(t)}^{q(t)} K(t, s, x(t), x(s))\, d_s \tau(t, s)) = C(t) \qquad \dots (9)$$

which satisfies (2) enjoys property (3).

The assertion of Theorem 6 follows from Theorem 1 since it is immediately verified that from condition 1 of Theorem 6 there follow conditions H6 and H7. If $x \in C([t_0, \infty); \mathbb{R})$, then $\mathcal{A} x \in C([T_{Ax}, \infty); \mathbb{R})$ (see Myshkis[3]). i.e. condition H5 is met too.

Corollary 3 — Let the following conditions hold :

1. Conditions 1 and 2 of Theorem 5 (conditions 1 and 2 of Theorem 6) are met.

2. Conditions 1, 2 and 3 of Corollary 1 are satisfied.

Then each oscillating solution x of eqn. (8) (of eqn. (9)) which satisfies (2) enjoys property (3).

Corollary 3 follows from Theorem 5 (Theorem 6) and Corollary 1.

Example 2 — Consider the differential equation :

$$[t^2 [t^2 [t^2 x'(t)]']']' + t^{-2} x^2 (t^3) = t^{-32}. \sin^2(\ln t)$$
$$+ 50 t^{-3} \sin(\ln t) - 140 t^{-3} \cos(\ln t), \quad t \geq 1.$$

Here $(\mathcal{A}x)(t) = x(t^3)$. Moreover, the functions $\tau_i(t) = t^2$ $(i = 1, 2, 3)$, $g(t) = t^3$, $F(t, u, v) = t^{-2} v^{-2}$ satisfy the conditions of Corollary 3 for $m(t) = k$, where k is a positive constant. Then each bounded oscillating solution x of the above equation enjoys property (3). For instance, $x(t) = \frac{\sin(\ln t)}{t^5}$ is a solution for which $\lim_{t \to \infty} (L_i x)(t) = 0$ $(i = 1, 2, 3)$

Corollary 4 — Let the following conditions hold :

1. Conditions 1 and 2 of Theorem 5 (conditions 1 and 2 of Theorem 6) are met.
2. Conditions 1, 2 and 3 of Corollary 2 are satisfied.

Then each oscillating solution x of eqn. (8) of eqn. (9) which satisfies (2) enjoys property (3).

Corollary 4 follows from Theorem 5 (Theorem 6) and Corollary 2.

Example 3 — Consider the differential equation

$$[e^t [e^{-t} [e^t \cdot x'(t)]']']' + \tan t \cdot e^{-2t} \int_{\frac{\pi}{2}}^{t} e^{2s-t} \cdot x(s)\, ds$$

$$= -6\, e^{-t} \cos t - 8\, e^{-t} \sin t - e^{-3t} \sin t, \quad t > \frac{\pi}{2}\cdot$$

Here $(\mathcal{A}\, x)(t) = \int_{\frac{\pi}{2}}^{t} e^{2s-t}\, x(s)\, ds$. Moreover, the functions $\tau_1(t) = \tau_3(t) = e^t$, $\tau_2(t) = e^{-t}$ $k(t, s, u, v) = e^{2s-t} u$, $F(t, u, v) = \tan t \cdot e^{-2t} \cdot v$ satisfy the conditions of Theorem 6 for $m(t) = k$, where k is a positive constant. Then each oscillating solution x for which $|\, x(t)\, | \le Ck$ eventually, enjoys property (3). For instance, $x(t) = \frac{\sin 2t}{e^{2t}}$ is an oscillating bounded solutions of the equation considered, which enjoys property (3).

Corollary 5 — Let the following conditions hold :

1. Condition 1 of Corollary 3 is met.
2. Conditions 2 and 3 of Theorem 2 are satisfied.

Then all solutions of eqn. (8) (of eqn. (9)) which satisfy (2) are nonoscillating.

Corollary 5 follows from Corollary 1 and Theorem 2.

Example 4 — Consider the differential equation

$$(L_{2n}\, x)(t) + t^{-5}\, x(t) = [1 + t^{-4}] \ln t + n, \quad t \ge 0$$

where $$(L_0 x)(t) = x(t), \quad (L_k x)(t) = t\, [(L_{k-1}\, x)(t)]', \quad 1 \le k \le 2n.$$

Here $(\mathcal{A}x)(t) = x(t)$. Moreover, the functions $\tau_1(t) = t$, $(1 \le i \le 2n)$, $F(t, u, v) = t^{-5} v$, $C(t) = (1 + t^{-4}) \ln t + n$ satisfy the conditions of Corollary 5. Then all solutions of the equation considered which satisfy (2) for $m(t) = t^k$, $k = 0, 1, 2, 3$, are nonoscillating. For instance, $x(t) = t \ln t$ is such a solution.

Corollary 6 — Let the following conditions hold :

1. Condition 1 of Corollary 3 is met.

2. Conditions 2, 3 and 4 of Theorem 3 are satisfied.

Then each oscillating solution of eqn. (8) [of eqn. (9)] enjoys property (3).

Corollary 6 follows from Corollary 3 and Theorem 3.

Corollary 7 — Let the following conditions hold :

1. Condition of 1 Corollary 3 is met.

2. Conditions 2 and 3 of Theorem 4 are satisfied.

Then all solutions of eqn. (8) [of eqn. (9)] are nonoscillating.

Corollary 7 follows from Corollary 3 and Theorem 4.

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