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ON THE RATE OF CONVERGENCE OF SSB OPERATOR AND SSK OPERATOR FOR FUNCTIONS OF BOUNDED VARIATION*

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In this paper we define SSB operator and SSK operator and study their pointwise estimates for functions of bounded variation on [– 1, 1].

1. INTRODUCTION

Recently, much effort has been done to investigate the convergence of linear positive operator for bounded variation. This idea stems from Bojanic and Vuillemier[2]. In this paper we first define two kinds of linear positive operators. i.e., SSB operator and SSK operator. Then we study the rate of convergence of such two kinds of linear positive operator for functions of bounded variation. One will find that it is obvious that they are of some generality.

Let

$$L_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n+\alpha_n}\right) q_{n,k,s}(x). \qquad \text{... (1.1)}$$

$$L_n^*(f,x) = (\varphi_n+1) \sum_{k=0}^{n} q_{n,k,s}(x) \int_{k/(\varphi_n+1)}^{(k+1)/(\varphi_n+1)} f(t)\,dt \qquad \text{... (1.2)}$$

where

$$q_{n,k,s}(x) = \begin{cases} (1-x)P_{n-s,k}(x), & 0 \le k < s \\ (1-x)P_{n-s,k}(x) + xP_{n-s,k-s}(x), & s \le k \le n-s \\ xP_{n-s,k-s}(x), & n-s < k \le n \end{cases} \qquad \text{... (1.3)}$$

$$P_{n,k} = \binom{n}{k} x^k (1-x)^{n-k} \qquad \text{... (1.4)}$$

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and $\varphi_n = n + \alpha_n,\ \alpha_n \geq 0$ satisfying

$$\lim_{n \to \infty} \frac{\alpha_n}{\sqrt{n}} = 0. \qquad \text{... (1.5)}$$

We call L_n and L_n^* as Stancu-Sikkema-Bernstein (SSB) operator and Stancu-Sikkema-Kantorovich (SSK) operator, respectively. Our main results are as follows :

Theorem 1.1 — Let f be the normalized function of bounded variation on [0,1]. That is, f is a function of bounded variation and $f(x) = \frac{1}{2}(f(x+0) + f(x-0))$ for all $x \in (0, 1)$. Then for any $x \in (0, 1)$ as n is large enough, it holds the following

$$\left| L_n(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right|$$

$$\leq \frac{30s + 13 + 10\alpha_n}{4\sqrt{(n-s)x(1-x)}} \left| f(x+) - f(x-) \right|$$

$$+ \frac{5}{nx(1-x)} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} (gx). \qquad \text{... (1.6)}$$

Theorem 1.2 — Let f be a normalized function of bounded variation on [0, 1]. Then for any $x \in (0, 1)$ as n is large enough, it holds the following :

$$\left| L_n^*(f, x) - \frac{1}{2}(f(x+) + f(x-)) \right|$$

$$\leq \frac{30s + 23 + 10\alpha_n}{4\sqrt{(n-s)x(1-x)}} \left| f(x+) - f(x-) \right|$$

$$+ \frac{5\, V_0^1(f)}{2\sqrt{(n-s)\, x(1-x)}} + \frac{5}{nx(1-x)} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + (1-x)/\sqrt{k}} (g_x) \qquad \text{... (1.7)}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+), & x < t \leq 1 \\ 0, & t = x \\ f(t) - f(x-), & 0 \leq t < x \end{cases} \qquad \text{... (1.8)}$$

and $V_b^a(g_x)$ denotes the total variation of $g_x(t)$ on $[a, b]$.

2. Lemmas

Lemma 2.1 — For any $0 \leq k \leq n, x \in (0, 1)$,

$$q_{n,k,s}(x) \le \frac{5}{2\sqrt{(n-s)x(1-x)}}. \qquad \text{... (2.1)}$$

PROOF : From Lemma 3 of Guo[2] we have

$$P_{n,k}(x) \le \frac{5}{2\sqrt{nx(1-x)}}. \qquad \text{... (2.2)}$$

From (1.3) and (2.2) we obtain (2.1).

Lemma 2.2 — For any $x \in (0, 1)$,

$$\left| \sum_{k > (n+\alpha_n)x} q_{n,k,s}(x) - \frac{1}{2} \right| \ge \frac{15s + 4 + 5\alpha_n}{2\sqrt{(n-s)x(1-x)}}. \qquad \text{... (2.3)}$$

PROOF : From Lemma 2 of Guo[2] we have

$$\left| \sum_{k > nx} P_{n,k}(x) - \frac{1}{2} \right| \le \frac{2}{\sqrt{nx(1-x)}}. \qquad \text{... (2.4)}$$

Since

$$\left| \sum_{k > nx} [(1-x) P_{n-s,k}(x) + xP_{n-s,k-s}(x)] - \frac{1}{2} \right|$$

$$\le (1-x) \left| \sum_{k > nx} P_{n-s,k}(x) - \frac{1}{2} \right| + x \left| \sum_{k > nx} P_{n-s,k-s}(x) - \frac{1}{2} \right|. \qquad \text{... (2.5)}$$

From (2.2) and (2.4) we have

$$\left| \sum_{k > nx} P_{n-s,k}(x) - \frac{1}{2} \right|$$

$$\le \left| \sum_{k > (n-s)x} P_{n-s,k}(x) - \frac{1}{2} \right| + \frac{5sx}{2\sqrt{(n-s)x(1-x)}}$$

$$\le \frac{2}{\sqrt{(n-s)x(1-x)}} + \frac{5sx}{2\sqrt{(n-s)x(1-x)}} \qquad \text{... (2.6)}$$

$$\left| \sum_{k > nx} P_{n-s,k-s}(x) - \frac{1}{2} \right|$$

$$\le \left| \sum_{k-s > (n-s)x} P_{n-s,k-s}(x) - \frac{1}{2} \right| + \frac{5s(1-x)}{2\sqrt{(n-s)x(1-x)}}$$

$$\le \frac{2}{\sqrt{(n-s)x(1-x)}} + \frac{5s(1-x)}{2\sqrt{(n-s)x(1-x)}}. \qquad \text{... (2.7)}$$

From (2.1), (2.5)-(2.7) and the definition of $q_{n,k,s}$ we have

$$\left|\sum_{k>nx} q_{n,k,s}(x)-\frac{1}{2}\right| \le 2s\frac{5}{2\sqrt{(n-s)\,x\,(1-x)}}$$

$$+\left|\sum_{k>nx}\left[(1-x)\,P_{n-s,k}(x)+xP_{n-s,k-s}(x)\right]-\frac{1}{2}\right|$$

$$\le \frac{15s+4}{2\sqrt{(n-s)x\,(1-x)}}. \qquad \dots (2.8)$$

From (2.1) and (2.8) we have

$$\left|\sum_{k>(n+\alpha_n)x} q_{n,k,s}(x)-\frac{1}{2}\right|$$

$$\le \left|\sum_{k>nx} q_{n,k,s}(x)-\frac{1}{2}\right|+\frac{5x\,\alpha_n}{2\sqrt{(n-s)\,x\,(1-x)}}$$

$$\le \frac{15s+4+5\alpha_n}{2\sqrt{(n-s)x\,(1-x)}}.$$

This completes the proof of Lemma 2.2.

Let

$$K_n(x,t) = \begin{cases}\sum\limits_{k\le(n+\alpha_n)t} q_{n,k,s}(x), & 0<t\le 1\\ 0, & t=0.\end{cases} \qquad \dots (2.9)$$

It is obvious that

$$L_n(f,x)=\int_0^1 f(t)\,d_t K_n(x,t). \qquad \dots (2.10)$$

Lemma 2.3 — As n is large enough,

(1) if $0\le y<x$, then

$$K_n(x,y)\le\frac{2x(1-x)}{n(x-y)^2}, \qquad \dots (2.11)$$

(2) if $x<z\le 1$, then

$$1-K_n(x,z)\le\frac{2x(1-x)}{n(x-z)^2}. \qquad \dots (2.12)$$

PROOF : From

$$L_n(1,x)=\sum_{k=0}^{n} q_{n,k,s}(x)$$

$$= (1-x)\sum_{k=0}^{n-s} P_{n-s,k}(x) + x\sum_{k=s}^{n} P_{n-s,k-s}(x)$$

$$= (1-x) + x = 1. \qquad \ldots (2.13)$$

$$L_n(t,x) = \sum_{k=0}^{n-s} \frac{k}{n+\alpha_n}(1-x)P_{n-s,k}(x)$$

$$+\sum_{k=s}^{n} \frac{k}{n+\alpha_n} x P_{n-s,k-s}(x) = \frac{nx}{n+\alpha_n}. \qquad \ldots (2.14)$$

By a series of computations we obtain that

$$L_n(t^2,x) = \frac{n^2x^2}{(n+\alpha_n)^2} + \frac{nx(1-x)}{(n+\alpha_n)^2} + \frac{s(1-s)x(1-x)}{(n+\alpha_n)^2}. \qquad \ldots (2.15)$$

Therefore, as n is large enough, from (2.13)-(2.15) we have

$$L_n((t-x)^2,x) = L_n(t^2,x) - 2x L_n(t,x) + x^2$$

$$= \frac{\alpha_n^2 x^2}{(n+\alpha_n)^2} + \frac{nx(1-x)}{(n+\alpha_n)^2} + \frac{s(1-s)x(1-x)}{(n+\alpha_n)^2} \qquad \ldots (2.16)$$

$$\le \frac{2x(1-x)}{n}.$$

If $0 \le y < x$, for any $t \in [0,y]$, we have $(x-t)/(x-y) \ge 1$. Therefore, from (2.10) and (2.16) we have

$$K_n(x,y) = \int_0^y d_t K_n(x,t) \le \int_0^y \left(\frac{x-t}{x-y}\right) d_t K_n(x,t)$$

$$= \frac{1}{(x-y)^2} L_n((t-x)^2,x) \le \frac{2x(1-x)}{n(x-y)^2}.$$

This completes the proof of (2.11).

The proof of (2.12) is similar. This completes the proof of Lemma 2.3.

3. Proof of Theorems

Proof of Theorem 1.1 — From

$$f(t) = \frac{1}{2}(f(x+) + f(x-)) + g_x(t) + \frac{1}{2}(f(x+) - f(x-)\,\mathrm{sgn}\,(t-x),$$

we have

$$\left| L_n(f,x) - \frac{1}{2}(f(x+) + f(x-)) \right|$$

$$\leq | L_n(g_x, x) | + \frac{1}{2} | f(x+) - f(x-) | \ | L_n(\text{sgn}(t-x), x) |. \quad \text{... (2.17)}$$

(1) The estimate of $| L_n(g_x x) |$: Let

$$I_1 = [0, x - x/\sqrt{n}], \ \ I_2 = [x - x/\sqrt{n}, x + (1-x)/\sqrt{n}]$$

$$I_3 = [x + (1-x)/\sqrt{n}, 1].$$

From (2.10) we have

$$L_n(f,x) = \int_0^1 f(t)\, d_t K_n(x,t) = \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) f(t)\, d_t K_n(x,t) \text{ tag} \quad \text{... (2.18)}$$

$$:= A_n(f,x) + B_n(f,x) + C_n(f,x).$$

Since the following process is similar to that of Bojanic and Vuillemier[1] we only give the main results and omit the details.

$$| B_n(f,x) | \leq V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x). \quad \text{... (2.19)}$$

$$| A_n(f,x) | \leq \frac{4(1-x)}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{n}}^{x} (g_x). \quad \text{... (2.20)}$$

$$| C_n(f,x) | \leq \frac{4x}{n(1-x)} V_{x}^{x+(1-x)/\sqrt{n}} (g_x). \quad \text{... (2.21)}$$

From (2.18)-(2.21) we have

$$| L_n(g_x, x) | \leq \frac{5}{nx(1-x)} \sum_{k=1}^{n} V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}} (g_x). \quad \text{... (2.22)}$$

(2) The estimate of $| L_n(\text{sgn}(t-x), x) |$: If for any $0 < k \leq n, x \neq k/(n+\alpha_n)$, then

$$| L_n(\text{sgn}(t-x), x) | \leq \left| \sum_{k > (n+\alpha_n)x} q_{n,k,s}(x) \right| - \left| \sum_{\frac{k}{n+\alpha_n} < x} q_{n,k,s}(x) \right| \quad \text{... (2.23)}$$

$$:= | F_n(x) - G_n(x) |.$$

Since $F_n(x) + G_n(x) = L_n(x) = 1$, from Lemma 2.2 we have

$$| L_n (\mathrm{sgn}\,(t-x), x) | \leq 2 \left| F_n(x) - \frac{1}{2} \right|$$

$$\leq \frac{15s + 4 + 5\alpha_n}{\sqrt{(n-s)x(1-x)}}. \quad \ldots (2.24)$$

If there exists a $k_0 > 0$ such that $x = k_0/(n+\alpha_n)$, then

$$F_n(x) + G_n(x) + q_{n, k_0, s}(x) = 1.$$

Therefore, from Lemma 2.1 and 2.2 we have

$$| L_n (\mathrm{sgn}\,(t-x), x) | \leq 2 \left| F_n(x) - \frac{1}{2} \right| + q_{n, k_0, s}(x)$$

$$\leq \frac{30s + 13 + 10\alpha_n}{2\sqrt{(n-s)x(1-x)}}. \quad \ldots (2.25)$$

From (2.24) and (2.25) we have

$$| L_n (\mathrm{sgn}\,(t-x), x) | \leq \frac{30s + 13 + 10\alpha_n}{2\sqrt{(n-s)x(1-x)}}. \quad \ldots (2.26)$$

From (2.17), (2.22) and (2.26) we obtain the proof of the Theorem 1.1.

The proof of the Theorem 1.2 is similar. We omit the details.

Remark 1 : When $s = 0$, $L_n(f, x)$ and $L_n^*(f, x)$ are just $C_n(f, x)$ and $C_n^*(f, x)$ of Guo[2]. Therefore, our results is the generalization of Guo[2].

Remark 2 : Our results are sharp in the sense of the order of convergence. The method of proof is standard and we omit the details.

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