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UNIQUENESS THEOREMS FOR MEROMORPHIC FUNCTIONS

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In this paper, we deal witth the problem of uniqueness of meromorphic functions, and answer a question posed by *F.* Gross[6]. It is shown that there exist three finite sets S_j (j = 1, 2, 3) such that any two nonconstant meromorphic functions f and g satisfying $Ef(S_j) = Eg(S_j)$ for j = 1, 2, 3 must be identical.

1. INTRODUCTION AND MAIN RESULTS

By a meromorphic function we shall always mean a meromorphic function in the complex plane. It is assumed that the reader is familiar with the notations of the Nevanlinna Theory (see, for example, Hayman[1]). We use E to denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ $(r \to \infty, \ r \notin E)$.

For any set S and any meromorphic function f let

$$E_f(S) = f^{-1}(S)$$

where we take due account of multiplicity; we also define

$$E_f(\{\infty\}) = E_F(\{0\})$$

where $F = 1/f$ (cf. Gross[2]).

Nevanlinna[3] proved the following well-known theorem.

Theorem $A^{3,4}$ — Let $S_j = \{a_j\}$ $(j = 1, 2, 3, 4)$, where a_1, a_2, a_3 and a_4 are four distinct complex numbers ($a_j = \infty$ is allowed). Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3, 4. Then either $f = g$, or f is a linear fractional transformation of g, two of the values, say a_1 and a_2, must be Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.

Using Theorem A, the present author proved that there exist four finite sets S_j (j = 1, 2, 3, 4) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3, 4 must be identical (see Yi[5]). Gross[6] also proved that there exists three finite sets S_j (j = 1, 2, 3) such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3, must be identical, and asked the following $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3 must be identical, and asked the following open question (see Gross[6], Question 6) :

Question 1 — Can one find two finite sets S_j (j = 1, 2) such that any two nonconstant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2 must be identical ?

Now it is natural to ask the following question :

Question 2 — Can one find three finite sets S_j (j = 1, 2, 3) such that any two nonconstant meromorphic functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3 must be identical ?

Throughout this paper we shall use w and u to denote the constants exp $(2\pi i/n)$ and exp $(2\pi i/m)$ respectively, where n and m are positive integers.

In this paper we answer the above questions. In fact, for Question 2, we prove more generally the following theorems.

Theorem 1 — Let $S_1 = \{a_1 + b_1, a_1 + b_1 w, \ldots, a_1 + b_1 w^{n-1}\}$, $S_2 = \{a_2 + b_2, a_2 + b_2 u, \ldots, a_2 + b_2 u^{m-1}\}$ and $S_3 = \{\infty\}$, where $n > 6$, $m > 6$, a_1, b_1, a_2 and b_2 are constants such that $b_1 b_2 \neq 0$ and $a_1 \neq a_2$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3. Then $f = g$.

Theorem 2 — Let $S_1 = \{a + b_1, a + b_1 w, \ldots, a + b_1 w^{n-1}\}$, $S_2 = \{a + b_2, a + b_2 u, \ldots, a + b_2 u^{m-1}\}$ and $S_3 = \{\infty\}$, where n (> 6) and m (> 6) have no common factor, and a, b_1 and b_2 are constants such that $b_1 b_2 \neq 0$ and $b_1^{2mn} \neq b_2^{2mn}$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3. Then $f = g$.

Theorem 3 — Let S_1 and S_2 be defined as in Theorem 2, and let $S_3 = \{a\}$. Suppose that f and g are nonconstant meromorphic functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2, 3. Then $f = g$.

From Theorem 1 and Theorem 2 we immediately obtain the following results, which answer Question 1 posed by Gross.

Theorem 4 — Let S_1 and S_2 be defined as in Theorem 1. Suppose that f and g are nonconstant entire functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2. Then $f = g$.

Theorem 5 — Let S_1 and S_2 be defined as in Theorem 2. Suppose that f and g are nonconstant entire functions satisfying $E_f(S_j) = E_g(S_j)$ for j = 1, 2. Then $f = g$.

Gross and Osgood[7] proved the following theorem.

Theorem B — Let $S_1 = \{-1, 1\}$, $S_2 = \{0\}$. If f and g are nonconstant entire functions of finite order such that $E_f(S_j) = E_g(S_j)$ (j = 1, 2), then $f = \pm g$ or $fg = \pm 1$.

In an earlier paper[8], we proved that in the preceding theorem the order restriction of f and g can be removed. In another paper[9], we proved the following result which is an extension of the above results.

Theorem C — Let $S_1 = \{a + b, \ a + bw, ..., a + bw^{n-1}\}$, $S_2 = \{a\}$ and $S_3 = \{\infty\}$, where $n > 1$, a and b ($\neq 0$) are constants. If f and g are nonconstant meromorphic functions such that $E_f(S_j) = E_g(S_j)$ (j = 1, 2, 3), then $f - a = t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) = s$, where $s^n = b^{2n}$.

In this paper we prove the following interesting results which are some improvements of the above theorems. These results will be needed in the proof of our theorems.

Theorem 6 — Let $S_1 = \{a + b, \ a + bw, ..., a + bw^{n-1}\}$, $S_2 = \{\infty\}$, where $n > 6$, a and b ($\neq 0$) are constants. If f and g are nonconstant meromorphic functions such that $E_f(S_j) = E_g(S_j)$ (j = 1, 2), then $f - a = t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) = s$, where $s^n = b^{2n}$.

Theorem 7 — Let S_1 be defined as in Theorem 6, and let $S_2 = \{a\}$. If f and g are nonconstant meromorphic functions such that $E_f(S_j) = E_g(S_j)$ (j = 1, 2), then $f - a = t(g - a)$, where $t^n = 1$, or $(f - a)(g - a) = s$, where $s^n = b^{2n}$.

2. Some Lemmas

Lemma 1 — Let f and g be two nonconstant meromorphic functions, and let c_1, c_2 and c_3 be three nonzero constants. If $c_1 f + c_2 g = c_3$, then

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f).$$

Proof : Bv the second fundamental theorem (see Hayman[1]), we have

$$T(r, f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \left(f - \frac{c_3}{c_1}\right)^{-1}\right) + \overline{N}(r, f) + S(r, f)$$

$$= \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f)$$

which proves Lemma 1.

Lemma 2[3, 10] — Let $f_1, f_2, ..., f_n$ be linearly independent meromorphic functions satisfying $\sum_{j=1}^{n} f_j = 1$. Then for k = 1, 2, ..., n we have

$$T(r, f_k) < \sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + N(r, f_k) + N(r, D) - \sum_{j=1}^{n} N(r, f_j)$$

$$- N\left(r, \frac{1}{D}\right) + o(T(r)) \quad (r \notin E),$$

where D denotes the Wronskian

$$D = \begin{vmatrix} f_1 & f_2 & \dots f_n \\ f_1' & f_2' & \dots f_n' \\ \dots & \dots & \dots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots f_n^{(n-1)} \end{vmatrix}$$

and $T(r)$ denotes the maximum of $T(r, f_j)$, $j = 1, 2, \dots, n$.

Lemma 3[11] — Let f_1, f_2 and f_3 be three meromorphic functions satisfying $\sum_{j=1}^{3} f_j = 1$, and let $g_1 = -f_3/f_2$, $g_2 = 1/f_2$ and $g_3 = -f_1/f_2$. If f_1, f_2 and f_3 are linearly independent, then g_1, g_2 and g_3 are linearly independent.

Lemma 4[12] — Let f be a nonconstant meromorphic function, and let $P(f)$ be a polynomial in f of the form

$$P(f) = a_0 f^n + a_1 f^{n-1} + \dots + a_{n-1} f + a_n,$$

where a_0 ($\neq 0$), $a_1, \dots, a_n$ are constants. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

3. Proof of Theorems 6 and 7

3.1. *Proof of Theorem* 6

Let $S_3 = \{1, w, \dots, w^{n-1}\}$, and let $F = (f - a)/b$ and $G = (g - a)/b$. By $E_f(S_j) = E_g(S_j)$ ($j = 1, 2$), we obtain $E_F(S_j) = E_G(S_j)$ ($j = 2, 3$). Then, from Nevanlinna's second fundamental theorem, we have

$$(n-1)\, T(r, G) < \sum_{k=0}^{n-1} N\left(r, \frac{1}{G - w^k}\right) + N(r, G) + S(r, G)$$

$$= \sum_{k=0}^{n-1} N\left(r, \frac{1}{F - w^k}\right) + N(r, F) + S(r, G)$$

$$< (n + 1)\, T(r, F) + S(r, G). \qquad \dots (1)$$

Thus

$$T(r, G) = O(T(r, F)) \quad (r \notin E). \qquad \dots (2)$$

Again by $E_F(S_j) = E_G(S_j)$ ($j = 2, 3$), we obtain

$$F^n - 1 = e^h (G^n - 1), \qquad \dots (3)$$

where h is an entire function. From (1) and (3), we have

$$T(r, e^h) = T\left(r, \frac{F^n - 1}{G^n - 1}\right)$$
$$< T(r, F^n) + T(r, G^n) + O(1)$$
$$< nT(r, F) + \frac{n(n+1)}{n-1} T(r, F) + S(r, F).$$

Thus

$$T(r, e^h) = O(T(r, F)) \qquad (r \notin E). \qquad \text{... (4)}$$

Let us put

$$f_1 = F^n, \qquad \text{... (5)}$$

$$f_2 = e^h, \qquad \text{... (6)}$$

$$f_3 = -e^h G^n, \qquad \text{... (7)}$$

and $T^*(r)$ denote the maximum of $T(r, f_j)$, $j = 1, 2, 3$. From (3), (5), (6) and (7), we obtain

$$\sum_{j=1}^{3} f_j = 1. \qquad \text{... (8)}$$

From (2), (4), (5), (6) and (7), we have

$$T^*(r) = O(T(r, F)) \quad (r \notin E). \qquad \text{... (9)}$$

We discuss the following three cases.

Case (a) : Suppose neither f_2 nor f_3 is constant.

We discuss the following two subcases.

(a_1) Assume that f_1, f_2 and f_3 are linearly independent. Applying Lemma 2 to functions f_j ($j = 1, 2, 3$), from (8) and (9) we have

$$T(r, f_1) < \sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) - N\left(r, \frac{1}{D}\right) + N(r, D) - N(r, f_2) - N(r, f_3)$$
$$+ o(T(r, F)) \quad (r \notin E), \qquad \text{... (10)}$$

where

$$D = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} \qquad \text{... (11)}$$

From (5), (6) and (7), we have

$$\sum_{j=1}^{3} N\left(r, \frac{1}{f_j}\right) = nN\left(r, \frac{1}{F}\right) + nN\left(r, \frac{1}{G}\right), \qquad \text{... (12)}$$

and, by looking at the zeros of F and G, from (5), (6), (7) and (11) we see that

$$N\left(r, \frac{1}{D}\right) \geq nN\left(r, \frac{1}{F}\right) - 2\overline{N}\left(r, \frac{1}{F}\right) + nN\left(r, \frac{1}{G}\right) - 2\overline{N}\left(r, \frac{1}{G}\right). \qquad \text{... (13)}$$

From (8) and (11) we get

$$D = \begin{vmatrix} f_2' & f_3' \\ f_2'' & f_3'' \end{vmatrix}. \qquad (14)$$

Since f_2 is entire, from (7) and (14) we have that

$$\begin{aligned} N(r, D) - N(r, f_2) - N(r, f_3) &= N(r, D) - N(r, f_3) \\ &\leq N(r, f_3'') - N(r, f_3) \\ &= N(r, (G^n)'') - N(r, G^n) \\ &= 2\overline{N}(r, G). \end{aligned} \qquad \text{... (15)}$$

From (5), (10), (12), (13) and (15) we deduce

$$\begin{aligned} nT(r, F) = T(r, f_1) &< 2\overline{N}\left(r, \frac{1}{F}\right) + 2\overline{N}\left(r, \frac{1}{G}\right) + 2\overline{N}(r, G) + S(r, F) \\ &< 2T(r, F) + 4T(r, G) + S(r, F). \end{aligned} \qquad \text{... (16)}$$

Let

$$g_1 = -f_3/f_2 = G^n, \qquad \text{... (17)}$$

$$g_2 = 1/f_2 = e^{-h}, \qquad \text{... (18)}$$

$$g_3 = -f_1/f_2 = -e^{-h} F^n. \qquad \text{... (19)}$$

From (8), (17), (18) and (19), we obtain

$$\sum_{j=1}^{n} g_j = 1.$$

By Lemma 3 we know that g_1, g_2 and g_3 are linearly independent. In the same manner as above, we have

$$nT(r, G) < 4T(r, F) + 2T(r, G) + S(r, F). \qquad \text{... (20)}$$

Combining (16) and (20) we get

$$(n - 6)\, T(r, F) + (n - 6)\, T(r, G) < S(r, F). \qquad \text{... (21)}$$

Since $n > 6$, (21) is a contradiction.

(a_2) Assume that f_1, f_2 and f_3 are linearly dependent. Then, there exist three constants $(c_1, c_2, c_3) \neq (0, 0, 0)$ such that

$$c_1 f_1 + c_2 f_2 + c_2 f_3 = 0. \quad \text{... (22)}$$

If $c_1 = 0$, from (22) we have $c_2 \neq 0$, $c_3 \neq 0$ and

$$f_3 = -\frac{c_2}{c_3} f_2 .$$

Hence, from (6) and (7) we obtain

$$G^n = \frac{c_2}{c_3} ,$$

which is impossible. Thus $c_1 \neq 0$ and

$$f_1 = -\frac{c_2}{c_1} f_2 - \frac{c_3}{c_1} f_3 . \quad \text{... (23)}$$

Now combining (8) and (23) we get

$$\left(1 - \frac{c_2}{c_1}\right) f_2 + \left(1 - \frac{c_3}{c_1}\right) f_3 = 1. \quad \text{... (24)}$$

Since neither f_2 nor f_3 is constant, we have $c_1 \neq c_2$ and $c_1 \neq c_3$. From (6), (7) and (24) we obtain

$$\left(1 - \frac{c_3}{c_1}\right) G^n + e^{-h} = 1 - \frac{c_2}{c_1} . \quad \text{... (25)}$$

By Lemma 1 and (25) we get

$$nT(r, G) < \overline{N}\left(r, \frac{1}{G}\right) + S(r, G)$$
$$< T(r, G) + S(r, G),$$

which is again a contradiction.

Case (b) : Suppose that $f_2 = c$ ($\neq 0$). If $c \neq 1$, from (8) we have

$$f_1 + f_3 = 1 - c.$$

Hence, from (5), (6) and (7) we obtain

$$F^n - cG^n = 1 - c. \quad \text{... (26)}$$

By Lemma 1 we have

$$nT(r, F) < \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + S(r, F)$$

$$< 2T(r, F) + T(r, G) + S(r, F),$$

and

$$nT(r, F) < \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, G) + S(r, G)$$

$$< 2T(r, F) + 2T(r, G) + S(r, G).$$

Hence,

$$(n - 3)T(r, F) + (n - 3)T(r, G) < S(r, F) + S(r, G),$$

which is impossible. Thus $c = 1$. From (26) we deduce $F^n = G^n$ and $F = tG$, where $t^n = 1$. Thus $f - a = t(g - a)$, where $t^n = 1$.

Case (c) : Suppose that $f_3 = c\ (\neq 0)$. If $c \neq 1$, from (8) we have

$$f_1 + f_2 = 1 - c.$$

Hence, from (5) and (6) we obtain

$$F^n + e^h = 1 - c. \qquad \text{... (27)}$$

By Lemma 1 we have

$$nT(r, F) < \overline{N}\left(r, \frac{1}{F}\right) + S(r, F)$$

$$< T(r, F) + S(r, F),$$

which is impossible. Thus $c = 1$. From (27) we have $F^n = -e^h$, $G^n = -e^{-h}$ and $F^n G^n = 1$. Thus $(f - a)\ (g - a) = s$, where $s^n = b^{2n}$.

This completes the proof of Theorem 6.

3.2. *Proof of Theorem* 7

Let $S_3 = \{\infty\}$, and let $F = a + \dfrac{b^2}{f-a}$, $G = a + \dfrac{b^2}{g-a}$. Since $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$, we obtain $E_F(S_j) = E_G(S_j)$ $(j = 1, 3)$. Applying Theorem 6 to meromorphic functions F and G, we have $F - a = t_1\ (G - a)$, where $t_1^n = 1$, or $(F-a)\ (G - a) = s_1$, where $s_1^n = b^{2n}$. Thus $f - a = t(g - a)$, where $t^n = (1/t_1)^n = 1$ or $(f-a)\ (g-a) = s$, where $s^n = (b^4/s_1)^n = b^{2n}$. This proves Theorem 7.

4. Proof of Theorems 1, 2 and 3

4.1 *Proof of Theorem* 1

By the assumption $E_f(S_j) = E_g(S_j)$ $(j = 1, 3)$, we have from Theorem 6

$$f - a_1 = t_1 (g - a_1), \qquad \text{... (28)}$$

where $t_1^n = 1$, or

$$(f - a_1)(g - a_1) = s_1, \qquad \text{... (29)}$$

where $s_1^n = b_1^{2n}$. In the same manner as above, by the assumption $E_f(S_j) = E_g(S_j)$ $(j = 2, 3)$, we have

$$f - a_2 = t_2 (g - a_2), \qquad \text{... (30)}$$

where $t_1^m = 1$, or

$$(f - a_2)(g - a_2) = s_2 , \qquad \text{... (31)}$$

where $s_2^m = b_2^{2m}$.

We discuss the following four cases.

Case (a) : Suppose that f and g satisfy (28) and (30). Then

$$a_2 - a_1 = (t_1 - t_2)g + (t_2 a_2 - t_1 a_1). \qquad \text{... (32)}$$

Since g is not a constant, and $a_1 \neq a_2$, we have from (32), $t_1 = t_2 = 1$. Thus $f = g$.

Case (b) : Suppose that f and g satisfy (28) and (31). Then

$$(t_1 g - t_1 a_1 + a_1 - a_2)(g - a_2) = s_2. \qquad \text{... (33)}$$

Applying Lemma 4 to (33), we have a contradiction.

Case (c) : Suppose that f and g satisfy (29) and (30). Similar to the case (b), we have again a contradiction.

Case (d) : Suppose that f and g satisfy (29) and (31). Then

$$(s_1 + (a_1 - a_2)(g - a_1))(g - a_2) = s_2 (g - a_1),$$

which contradicts Lemma 4.

This completes the proof of Theorem 1.

4.2. *Proof of Theorem* 2

By the assumption $E_f(S_j) = E_g(S_j)$ $(j = 1, 3)$, we have from Theorem 6

$$f - a = t_1 (g - a), \qquad \text{... (34)}$$

where $t_1^n = 1$, or

$$(f - a)(g - a) = s_1 , \qquad \text{... (35)}$$

where $s_1^n = b_1^{2n}$. In the same manner as above, by the assumption $E_f(S_j) = E_g(S_j)$ (j = 2, 3), we have

$$f - a = t_2 (g - a), \qquad \text{... (36)}$$

where $t_2^m = 1$, or

$$(f - a)(g - a) = s_2 , \qquad \text{... (37)}$$

where $s_2^m = b_2^{2m}$.

We discuss the following four cases.

Case (a) : Suppose that f and g satisfy (34) and (36). Then $t_1 = t_2$. Note that $t_1^n = 1$, $t_2^m = 1$, and n and m have no common factor. Thus $t_1 = t_2 = 1$, and hence $f = g$.

Case (b) : Suppose that f and g satisfy (34) and (37). Then

$$t_1 (g - a)^2 = s_2 ,$$

which is impossible.

Case (c) : Suppose that f and g satisfy (35) and (36). Similar to the case (b), we have again a contradiction.

Case (d) : Suppose that f and g satisfy (35) and (37). Then $s_1 = s_2$. Note that $s_1^n = b_1^{2n}$ and $s_2^m = b_2^{2m}$. Thus $b_1^{2mn} = b_2^{2mn}$, which contradicts the assumption.

This completes the proof of Theorem 2.

4.3. *Proof of Theorem* 3

Let $S_4 = \{a + b_2, a + b_2 w, \ldots, a + b_2 w^{n-1}\}$, $S_5 = \{a + b_1, a + b_1 u, \ldots, a + b_1 u^{m-1}\}$ and $S_6 = \{\infty\}$, and let $F = a + \dfrac{b_1 b_2}{f - a}$ and $G = a + \dfrac{b_1 b_2}{g - a}$. By the assumption $E_f(S_j) = E_g(S_j)$ (j = 1, 2, 3), it is easy to see that $E_F(S_j) = E_G(S_j)$ (j = 4, 5, 6). Applying Theorem 2 to meromorphic functions F and G, we have $F = G$. Thus $f = g$, which proves Theorem 3.

REFERENCES

1. W. K. Hayman, *Meromorphic Functions*, Clearendon Press, Oxford, 1964.
2. F. Gross, *Trans. Am. Math. Soc.* **131** (1968), 199-214.
3. R. Nevanlinna, *Le Theorems de Picard-Borel et la Theorie des Fonctions Meromorphes*, Gauthier-Villars, Paris, 1929.
4. G. G. Gundersen, *Trans. Am. Math. Soc.* **277** (1983), 545-67.
5. Hong-Xun Yi, *On a Result of Gross*, Shandong University Press, Jinan, 1990.
6. F. Gross, Factorization of meromorphic functions and some open problems, Complex Analysis (Proc. Conf. Univ. Kentucky, Lexington, Ky. 1976), *Lecture Notes in Math.* Vol. 599, Springer, Berlin, 1977, pp. 51-69.
7. F. Gross and C. F. Osgood, Entire functions with common preimages, in : *Factorization Theory of Meromorphic Functions*, Marcel Dekker, Inc., 1982, pp. 19-24.

8. Hon-Xun Yi, *J. Math. (PRC)*, **7** (1987), 219-24.

9. Hong-Xun Yi, *Acta Math. Sin.* **31** (1988), 570-76.

10. F. Gross, *Factorization of Meromorphic Functions*, U. S. Govt. Printing Office Publication, Washington, D. C., 1972.

11. Hon-Xun Yi, *Kodai Math. J.* **13** : 3(1990), 363-72.

12. C. C. Yang, *Math. Z.* **125** (1972), 107-12.