Indian J. pure appl. Math., 25(7) : 755-758, July 1994

ALMOST CONVERGENCE AND A THEOREM OF LORENTZ

IVOR J. MADDOX

Department of Pure Mathematics, Queen's University of Belfast Belfast BT 7 1NN, United Kingdom

(*Received* 24 *September* 1993; *accepted* 24 *March* 1994)

Dedicated to the memory of Professor Brian Kuttner (1908-1992)

A theorem of Lorentz on almost convergence f and Toeplitz matrices A is extended to more general matrices. Also, for non-negative matrices A, a necessary and sufficient condition is proved for the inclusion $f_A \subseteq f$.

In what follows we suppose that $A = (a_{nk})$ is a real infinite matrix such that

$$\sum_{k=0}^{\infty} |a_{nk}| < \infty \text{ for each } n \geq 0. \qquad \text{... (1)}$$

By $x = (x_k)$ we denote a bounded real sequence, with $||x||$ defined to be $\sup_k |x_k| < \infty$.

Hence the series

$$A(n, i, x) := \sum_{k=0}^{\infty} a_{nk} x_{k+i}$$

converge absolutely for all $n \geq 0$ and all $i \geq 0$.

The space f_A is defined as the set of all bounded x for which there exists $s = s(x) \in R$ such that

$$A(n, i, x) \to s \ (n \to \infty, \text{ uniformly for } i \geq 0). \qquad \text{... (2)}$$

In case A is the $(C, 1)$ matrix of arithmetic means, we define $f_A = f$. Thus $x \in f$ if and only if there exists $t = t(x) \in R$ such that

$$(n+1)^{-1} \sum_{k=0}^{n} x_{k+i} \to t \quad (n \to \infty, \text{ uniformly for } i \geq 0). \qquad \text{... (3)}$$

If $x \in f_A$ and (2) holds then we say that x is f_A summable to s, written $x_k \to s(f_A)$.

Likewise, we write $x_k \to t(f)$, when A is the $(C, 1)$ matrix, and say that x is almost convergent to t when (3) holds.

For Toeplitz matrices A these ideas were introduced by Lorentz[2] who proved the following interesting relation between f_A and f :

Theorem 1 — If A is a real Toeplitz matrix then $x_k \to s(f_A)$ implies $x_k \to s(f)$.

In the present note we give a variant of Lorentz's result in which we consider much more general matrices A.

Our principal result is :

Theorem 2 — Let A be such that (1) holds. Then (a) implies (b) and (b) implies (c), where

(a) $\limsup_n \left| \sum_{k=0}^{\infty} a_{nk} \right| > 0$

(b) $f_A \subseteq f$

(c) $\limsup_n \sum_{k=0}^{\infty} | a_{nk} | > 0.$

PROOF : Let (a) hold and take any $x \in f_A$. Then there exists s such that (2) holds. Now choose any Banach limit L. Then it follows from the argument of Lorentz[2] that

$$L(x) \sum_{k=0}^{\infty} a_{nk} \to s \quad (n \to \infty). \qquad \text{... (4)}$$

We remark that (4) holds under the weaker condition (1) on A, since Lorentz does not in fact use the full force of his assumption that A is a Toeplitz matrix.

We now consider the logical possibilities. It could happen that $L(x) = 0$ for all Banach limits L, in which case it follows that $x_k \to O(f)$ (by Lorentz[2], Theorem 1), whence $x \in f$.

Otherwise, there exists a Banach limit L_0, say, such that $L_0(x)$ is nonzero. Then, applying (4) with L_0 in place of L we see that

$$\sum_{k=0}^{\infty} a_{nk} \to s/L_0(x) \quad (n \to \infty). \qquad \text{... (5)}$$

In view of (a) it follows from (5) that s is nonzero. Hence by (4), with an arbitrary Banach limit L we have

$$L(x) \ (s/L_0(x)) = s,$$

which implies, since s is nonzero, that $L(x) = L_0(x)$. Since this holds for every Banach limit L it follows that $x_k \to L_0(x)(f)$, whence $x \in f$. Consequently, (b) is valid.

Now let (b) hold but suppose, if possible, that (c) is false.

Then

$$\sum_{k=0}^{\infty} |a_{nk}| \rightarrow O(n \to \infty). \qquad \text{... (6)}$$

Let us now choose a bounded sequence x which is not almost convergent. For example, we could define such an x by

$x_k = 0$ for $0 \le k < 4$, and for $r = 2, 3, 4, \ldots$ define $x_k = 1(2^r \le k \le r + 2^r)$, and $x_k = O(r + 2^r < k < 2^{r+1})$. Hence $||x|| = 1$.

By (6), for all $n \ge 0$ and all $i \ge 0$ we have

$$\left| \sum_{k=0}^{\infty} a_{nk} x_{k+i} \right| \le \sum_{k=0}^{\infty} |a_{nk}| \cdot ||x|| \rightarrow 0,$$

as $n \to \infty$, uniformly for $i \ge 0$, whence $x \in f_A$, so by (b) we must have $x \in f$ contrary to the choice of x. Hence (b) implies (c), and the proof is complete.

For non-negative A we are able now to present a necessary and sufficient condition for the inclusion $f_A \subseteq f$:

Theorem 3 — Let $A = (a_{nk})$ be a non-negative matrix such that

$$\sum_{k=0}^{\infty} a_{nk} \text{ converges for all } n \ge 0.$$

Then $f_A \subseteq f$ if and only if

$$\limsup_n \sum_{k=0}^{\infty} a_{nk} > 0.$$

PROOF : This follows immediately from Theorem 2 since (a) and (c) are equivalent for non-negative A.

Finally, we shall prove that condition (c) of Theorem 2, which is necessary for the inclusion $f_A \subseteq f$ is not in general sufficient.

Theorem 4 — There exists a matrix A with

$$0 < \limsup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty \qquad \text{... (7)}$$

such that the inclusion $f_A \subseteq f$ is false.

PROOF : Let us define $A = (a_{nk})$ by taking $a_{n,n} = 1$ and $a_{n,n+1} = -1$, and $a_{nk} = 0$ otherwise. Then (7) holds since the upper limit in (7) is equal to 2. Now define the bounded sequence x by $x_0 = 0$ and $x_n = \cos(\log n)$ for all $n \ge 1$, where the logarithms are to be taken to the base 10. Then x is divergent since it contains the divergent subsequence

$$(\cos n) = (\cos 0, \cos 1, \cos 2, \ldots).$$

Also, by applying the mean value theorem to the function

$$f(t) = \cos(\log t)$$

we find that $n \mid x_n - x_{n-1} \mid \leq 1$ for all $n \geq 1$. Hence, defining $b_0 = x_0$, $b_n = x_n - x_{n-1}$ for all $n \geq 1$, we see that the series

$$b_0 + b_1 + b_2 + \ldots$$

is divergent and such that $n \mid b_n \mid \leq 1$ for all $n \geq 0$.

Now for the sequence x just constructed we have, for all $n \geq 1$ and for all $i \geq 0$,

$$\left| \sum_{k=0}^{\infty} a_{nk} x_{k+i} \right| = \left| x_{n+i} - x_{n+i+1} \right| \leq 1/(n+i) \leq 1/n$$

whence $x_k \to O(f_A)$.

Assume, if possible, that $x \in f$. Then it follows that x is $(C, 1)$ summable, whence the series $b_0 + b_1 + \ldots$ is $(C, 1)$ summable and such that (nb_n) is bounded. Hence, by a well-known Tauberian theorem due to Hardy (see for example Hardy[1], Theorem 63); the series Σb_n would have to be convergent, contrary to the fact that it is divergent by our construction. This completes the proof.

REFERENCES

1. G. H. Hardy, *Divergent Series*, Oxford University Press, Oxford, 1956.
2. G. G. Lorentz, *Acta. Math.* **80** (1948), 167-90.