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A GENERATING SOLUTION OF THE SMALL ECCENTRIC RESTRICTED THREE-BODY PROBLEM IN *KS*-VARIABLES

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In this paper a new form of generating solution has been established in the restricted problem of three bodies in a four-dimensional phase space for small value of eccentricity.

INTRODUCTION

Krasinski[2] established the generating solutions in the circular restricted problem of two bodies in planar case. Ahmad and Huda[1] generalized the problem in the circular restricted problem of three bodies in four-dimensional linear space dealing with some particular cases. For example they have chosen the inclination of the orbital plane as right angle and also they took 8 as the constant of integration of the energy integral without any justification. In this paper, we have presented a generating solution of the small eccentric restricted three-body problem in terms of *KS*-variables without any suitable choosing.

Here the orbital elements, introduced in generalised form are ω, J, Ω and ϕ where ω is the argument of the perihelion, J the inclination of the orbital plane of the infinitesir al mass with the equatorial plane, Ω the longitude of the ascending node, and ϕ th , true anomaly.

First three elements are arbitrary constants and ϕ is only the variable in this problem. *KS*-variables are used here also, which transform three dimensional physical space into four-dimensional linear space of real numbers.

EQUATIONS OF MOTION

If μ be the ratio of the mass of the smaller primary to the total mass of the primaries, then the regularised canonical equations of motion of the infinitesimal mass given by Kurcheeva[4] are

$$\frac{dq_i}{ds} = \frac{\partial K}{\partial Q_i}$$

$$\frac{dQ_i}{ds} = -\frac{\partial K}{\partial q_i} \qquad (i = 1, 2, 3, 4) \qquad \ldots (1)$$

where q_i's are the generalised co-ordinates and Q_i's are the corresponding generalized components of momenta with the Hamiltonian K, given by

$$K = \frac{1}{2} \sum_{i=1}^{4} Q_i^2 + 2\rho^2 [Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3] - 4\ (1 - \mu)$$

$$+\ 4\rho^2 \left[-C/2 - \frac{\mu^2}{2} - \frac{\mu}{r_2} + \mu \sum_{i=1}^{4} (-1)^{i+1} q_i^2 \right]. \qquad \ldots (2)$$

The distance between the infinitesimal mass and the bigger primary is given by

$$r_1 = \sum_{i=1}^{4} q_i^2 = \rho^2 . \qquad \ldots (3)$$

The distance between the infinitesimal mass and the smaller primary is given by

$$r_2^2 = 1 - 2 \sum_{i=1}^{4} (-1)^{i+1} \ q_i^2 + \left[\sum_{i=1}^{4} q_i^2 \right]^2 . \qquad \ldots (4)$$

The physical time t and fictitious time s are connected by the relation

$$dt = 4\rho^2 ds. \qquad \ldots (5)$$

For generating solution, $\mu = 0$, then from (2)

$$K = \frac{1}{2} \sum_{i=1}^{4} Q_i^2 + 2\rho^2 [Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3 + C] - 4. \qquad \ldots (6)$$

With the help of (1) and (6) the followings are obtained

$$\dot{q}_1 = Q_1 + 2\rho^2 q_2 , \quad Q_1 = \dot{q}_1 - 2\rho^2 q_2 ,$$

$$\dot{q}_2 = Q_2 - 2\rho^2 q_1 , \quad Q_2 = \dot{q}_2 + 2\rho^2 q_1 , \qquad \left(\cdot = \frac{d}{ds} \right)$$

$$\dot{q}_3 = Q_3 - 2\rho^2 q_4 , \quad Q_3 = \dot{q}_3 + 2\rho^2 q_4 ,$$

$$\dot{q}_4 = Q_4 + 2\rho^2 q_3 , \quad Q_4 = \dot{q}_4 - 2\rho^2 q_3 . \qquad \ldots (7)$$

Again with the help of (1), (3), (6) and (7) one can easily show that

$$\ddot{q}_1 - 4\rho^2 \dot{q}_2 - 4\,(q_1^2 + q_2^2)\,\dot{q}_2 - 4(q_2\,q_3 + q_1\,q_4)\,\dot{q}_3 - 4(q_2\,q_4 - q_1\,q_3)\,\dot{q}_4$$
$$= 4q_1\,(3\rho^4 - C) \qquad \text{... (8)}$$

$$\ddot{q}_2 + 4\rho^2 \dot{q}_1 + 4\,(q_1^2 + q_2^2)\,\dot{q}_1 + 4(q_1\,q_3 - q_2\,q_4)\,\dot{q}_3 + 4(q_1\,q_4 + q_2\,q_3)\,\dot{q}_4$$
$$= 4q_2\,(3\rho^4 - C) \qquad \text{... (9)}$$

$$\ddot{q}_3 + 4\rho^2 \dot{q}_4 + 4(q_1\,q_4 + q_2\,q_3)\,\dot{q}_1 + 4(q_2\,q_4 - q_1\,q_3)\,\dot{q}_2 + 4(q_3^2 + q_4^2)\,\dot{q}_4$$
$$= 4q_3\,(3\rho^4 - C) \qquad \text{... (10)}$$

$$\ddot{q}_4 - 4\rho^2 \dot{q}_3 - 4(q_1\,q_3 - q_2\,q_4)\,\dot{q}_1 - 4(q_2\,q_3 + q_1\,q_4)\,\dot{q}_2 - 4(q_3^2 + q_4^2)\,\dot{q}_3$$
$$= 4q_4\,(3\rho^4 - C). \qquad \text{... (11)}$$

Multiplying (8), (9), (10) and (11) by q_2, q_1, q_4 and q_3 respectively and subtracting the sum of 2nd and 3rd from the sum of the 1st and 4th and on integration one can easily find that

$$\dot{q}_1\,q_2 - q_1\,\dot{q}_2 - \dot{q}_3\,q_4 + q_3\,\dot{q}_4 = 2\rho^4 + \lambda. \qquad \text{... (12)}$$

Again multiplying eqns. (8), (9), (10) and (11) by $\dot{q}_1, \dot{q}_2, \dot{q}_3$ and $\dot{q}_4$ respectively and integrating their sum, we get

$$\sum_{i=1}^{4} \dot{q}_i^2 = 4\rho^6 - 4C\rho^2 + \nu \qquad \text{... (13)}$$

where λ and ν are the constants of integration depending upon the initial conditions.

In terms of arbitrary constant orbital elements, ω, J, Ω and the variable true anomaly ϕ, the parametric representation of *KS*-variables are given by (Stiefel and Scheifele[5], p. 68)

$$q_1 = \rho \sin J/2 \cos\left(\frac{\Omega - \omega - \phi}{2}\right)$$

$$q_2 = \rho \sin J/2 \sin\left(\frac{\Omega - \omega - \phi}{2}\right)$$

$$q_3 = \rho \cos J/2 \sin\left(\frac{\Omega + \omega + \phi}{2}\right)$$

$$q_4 = -\rho \cos J/2 \cos\left(\frac{\Omega + \omega + \phi}{2}\right) \qquad \text{... (14)}$$

which satisfy the relations

$$r_1 = \rho^2 = \sum_{i=1}^{4} q_i^2 = \frac{a(1-e^2)}{1+e\cos\phi} ,$$

$$r_2^2 = 1 - 2\sum_{i=1}^{4} (-1)^{i+1} q_i^2 + \left[\sum_{i=1}^{4} q_i^2\right]^2 \qquad \text{... (15)}$$

$$= 1 - 2\rho^2 [\sin\Omega \sin(\omega+\phi) \cos J \cos\Omega \cos(\omega+\phi)] + \rho^4 \qquad \text{... (16)}$$

and $$dt = 4\rho^2\, ds = \frac{4a(1-e^2)}{(1+e\cos\phi)}\, ds , \qquad \text{... (17)}$$

where e is the eccentricity of the elliptic orbit and a is the semimajor axis of the orbit.

Differentiating q_i's with respect to s, we get

$$\dot{q}_1 = \dot{\rho} \sin J/2 \cos\left(\frac{\Omega-\omega-\phi}{2}\right) + \frac{\dot{\phi}}{2}\rho \sin J/2 \sin\left(\frac{\Omega-\omega-\phi}{2}\right)$$

$$\dot{q}_2 = \dot{\rho} \sin J/2 \sin\left(\frac{\Omega-\omega-\phi}{2}\right) - \frac{\dot{\phi}}{2}\rho \sin J/2 \cos\left(\frac{\Omega-\omega-\phi}{2}\right) \qquad \text{... (18)}$$

$$\dot{q}_3 = \dot{\rho} \cos J/2 \sin\left(\frac{\Omega+\omega+\phi}{2}\right) + \frac{\dot{\phi}}{2}\rho \cos J/2 \cos\left(\frac{\Omega+\omega+\phi}{2}\right)$$

$$\dot{q}_4 = -\dot{\rho} \cos J/2 \cos\left(\frac{\Omega+\omega+\phi}{2}\right) + \frac{\dot{\phi}}{2}\rho \cos J/2 \sin\left(\frac{\Omega+\omega+\phi}{2}\right)$$

which satisfy the relations

$$\dot{q}_1 q_2 - q_1 \dot{q}_2 - \dot{q}_3 q_4 + q_3 \dot{q}_4 = \frac{1}{2}\rho^2 \dot{\phi}, \qquad \text{... (19)}$$

and $$\sum_{i=1}^{4} \dot{q}_1^2 = \dot{\rho}^2 + \frac{1}{4}\rho^2 \dot{\phi}^2 . \qquad \text{... (20)}$$

The combinations {(12), (15) and (19)} and {(13), (15) and (20)} yield

$$\frac{a(1-e^2)}{2(1+e\cos\phi)}\dot{\phi} = \frac{2a^2(1-e^2)^2}{(1+e\cos\phi)^2} + \lambda \qquad \text{... (21)}$$

$$\frac{a(1-e^2)\, e^2 \sin^2\phi}{4(1+e\cos\phi)^3}\dot{\phi}^2 + \frac{a(1-e^2)}{4(1+e\cos\phi)}\dot{\phi}$$
$$= \frac{4a^3(1-e^2)^3}{(1+e\cos\phi)^3} - \frac{4aC(1-e^2)}{1+e\cos\phi} + \nu. \qquad \text{... (22)}$$

Initially at the pericentre passage, i.e., at $s = 0$ and $\phi = 0$, $\dot{\phi} = 8\,\overline{\omega}\sqrt{\frac{1+e}{1-e}}$, then from eqn. (21) we get $\lambda = 4a\bar{\omega}\sqrt{1-e^2} - 2a^2(1-e)^2$, where the angular frequency is given by $\bar{\omega} = \dfrac{\bar{K}}{2\sqrt{a}}$ ($\bar{K}$ is the constant of gravitation). Therefore, eqn. (21) reduces to

$$\frac{a(1-e^2)}{2(1+e\cos\phi)}\dot{\phi} = \frac{2a^2(1-e^2)^2}{(1+e\cos\phi)^2} + 4a\bar{\omega}\sqrt{1-e^2} - 2a^2(1-e)^2.$$

For small value of e, neglecting the second and higher order terms of e, the final form of the above equation is

$$\dot{\phi} = x + y\cos\phi, \qquad \text{... (23)}$$

where $x = 8\,(ae + \bar{\omega})$, $y = 8e(a - \bar{\omega})$ are constants.

Here clearly $x \neq y$, then either $x > y$ or $x < y$.

Case I — When $x > y$, then eqn. (23) yields

$$\phi = 2\tan^{-1}\left[p\tan\left(\frac{s}{2}\sqrt{(x^2-y^2)} + pp_0\right)\right] \qquad \text{... (24)}$$

where $p = \sqrt{\dfrac{x+y}{x-y}}$, p_0 is the constant of integration.

Case II — When $x < y$, then eqn. (23) yields

$$\phi = 2\tan^{-1}\left[q\tanh\left(\frac{s}{2}\sqrt{(y^2-x^2)} + qq_0\right)\right] \qquad \text{... (25)}$$

provided $\tan\phi/2 < q = \sqrt{\dfrac{y+x}{y-x}}$,

and $$\phi = 2\tan^{-1}\left[-q\coth\left(qq_0 - \frac{s}{2}\sqrt{(y^2-x^2)}\right)\right] \qquad \text{... (26)}$$

provided $\tan\phi/2 < q = \sqrt{\dfrac{y+x}{y-x}}$,

q_0 is the constant of integration. Again initially at the pericentre passage, i.e., at $s = 0$ and $\phi = 0$, $\dot{\phi} = 8\bar{\omega}\,[(1+e)/(1-e)]^{1/2}$, eqn. (22) gives $\nu = 2a\bar{\omega}\sqrt{(1-e^2)} - 4a^3(1-e)^3 + 4aC(1-e)$.

Therefore eqn. (22) reduces to

$$\frac{a(1-e^2)\,e^2\sin^2\phi}{4(1+e\cos\phi)}\dot{\phi}^2 + \frac{a(1-e^2)}{4(1+e\cos\phi)}\dot{\phi} = \frac{4a^3(1-e^2)^3}{(1+e\cos\phi)^3} - \frac{4aC((1-e^2)}{(1+e\cos\phi)}$$
$$+\ 2a\bar{\omega}\sqrt{1-e^2} - 4a^3(1-e)^3 + 4aC(1-e).$$

For small value of e, neglecting the second and higher order terms of e, the final form of the above equation is

$$\dot{\phi} = l - m \cos \phi \quad \text{... (27)}$$

where $l = 8\ (\overline{\omega} + 6a^2 e - 2Ce)$ and $m = 8e(6a^2 - \overline{\omega} - 2C)$ are constants.

Here clearly $l \neq m$, then either $l > m$ or $l < m$.

Case III — When $l > m$, then eqn. (27) yields

$$\phi = 2 \tan^{-1} \left[\alpha \tan \left(\frac{s}{2} \sqrt{(l^2 - m^2)} + \alpha\, \alpha_0 \right) \right] \quad \text{... (28)}$$

where $\alpha = \sqrt{\dfrac{1-m}{1+m}}$ and α_0 is the constant of integration.

Case IV — Wherc $l < m$, then from cqn. (27) we get

$$\phi = 2 \tan^{-1} \left[-\beta \coth \left(\frac{s}{2} \sqrt{(m^2 - l^2)} + \beta\beta_0 \right) \right] \quad \text{... (29)}$$

provided $\tan \phi/2 > \beta = \sqrt{\dfrac{m-l}{m+l}}$

and $$\phi = 2 \tan^{-1} \left[\beta \tanh \left(\beta\, \beta_0 - \frac{s}{2} \sqrt{m^2 - l^2} \right) \right] \quad \text{... (30)}$$

provided $\tan \phi/2 < \beta = \sqrt{\dfrac{m-l}{m+l}}$, β_0 is the constant of integration.

Among the above six solutions, viz. (24), (25), (26), (28), (29) and (30) none is found suitable for showing the analytic periodic solution of the motion of the infinitesimal mass. Therefore, it may be possible that there will be some other integrals, which will produce periodic solution. However, without numerical verification it cannot be justified that which one will produce the periodic solution.

From (17) the relation between the physical time t and fictitious time s can be written as

$$t = 4a\,(1 - e^2) \int \frac{ds}{1 + e\cos \phi} \cdot \quad \text{... (31)}$$

On substituting the value of ϕ from (24), (25), (26), (28), (29) and (30) in (31) one can integrate numerically, as analytical integration is not possible for (31).

Therefore the generating solutions are

$$q_1 = \rho \sin J/2 \cos \left(\frac{\Omega - \omega - \phi}{2} \right), \quad q_2 = \rho \sin J/2 \sin \left(\frac{\Omega - \omega - \phi}{2} \right)$$

$$q_3 = \rho \cos J/2 \sin \left(\frac{\Omega + \omega + \phi}{2} \right), \quad q_4 = -\rho \cos J/2 \cos \left(\frac{\Omega + \omega + \phi}{2} \right)$$

$$Q_1 = \frac{\sin J/2\sqrt{a(1-e^2)}}{2(1+e\cos\phi)^{3/2}}$$

$$\times\left[e\dot{\phi}\cos\left(\frac{\Omega-\omega+\phi}{2}\right)+\{\dot{\phi}-4a(1-e^2)\}\sin\left(\frac{\Omega-\omega-\phi}{2}\right)\right]$$

$$Q_2 = \frac{\sin J/2\sqrt{a(1-e^2)}}{2(1+e\cos\phi)^{3/2}}$$

$$\times\left[e\dot{\phi}\cos\left(\frac{\Omega-\omega+\phi}{2}\right)+\{\dot{\phi}-4a(1-e^2)\}\cos\left(\frac{\Omega-\omega-\phi}{2}\right)\right]$$

$$Q_3 = \frac{\cos J/2\sqrt{a(1-e^2)}}{2(1+e\cos\phi)^{3/2}}$$

$$\times\left[e\dot{\phi}\cos\left(\frac{\Omega+\omega-\phi}{2}\right)+\{\dot{\phi}-4a(1-e^2)\}\cos\left(\frac{\Omega+\omega+\phi}{2}\right)\right]$$

$$Q_4 = \frac{\cos J/2\sqrt{a(1-e^2)}}{2(1+e\cos\phi)^{3/2}}$$

$$\times\left[e\dot{\phi}\sin\left(\frac{\Omega+\omega-\phi}{2}\right)+\{\dot{\phi}-4a(1-e^2)\}\sin\left(\frac{\Omega+\omega+\phi}{2}\right)\right] \quad \ldots (32)$$

together with eqns. (24), (25), (26), (28), (29), (30) and (31).

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REFERENCES

1. A. Ahmad and M.N. Huda, *Bull. ASI* **14**, (1986), 11.
2. G. A. Krasinski, *Bull. Inst. Theor. Astron.* **9**, (1963), 105
3. P. Kustaanheimo, and E. L. Stiefel, *J. Reine Angew. Math.* **218**, (1965), 204.
4. I. V. Kurcheeva, *Cel. Mech.* **15** (1977), 353
5. E. L. Stiefel and G. Scheifele, *Linear and Regular Celestial Mechanics*, Springer-Verlag, Berlin, Heidlberg, New York, 1971.