Indian J. pure appl. Math., 25(8) : 801-813, August 1994

SUFFICIENT OPTIMALITY CRITERIA AND DUALITY FOR MULTIOBJECTIVE VARIATIONAL PROBLEMS WITH *V*-INVEXITY

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(*Received* 16 *August* 1993; *after revision* 24 *January* 1994; *accepted* 8 *April* 1994)

The sufficient optimality criteria for a class of multiobjective variational problems under *V*-invexity assumption is presented. Duality results are proved under a variety of *V*-invexity assumptions. These results prove many of the duality theorems of variational problems under weaker assumptions.

1. INTRODUCTION

Hanson[2] extended the Wolfe-duality results of mathematical programming to a class of functions subsequently called invex functions. Since then many researchers tried to relax the convexity assumptions further (e.g., Preda[8], Zalmai[10] and recently Jeyakumar and Mond[3]). It has been shown[4-7, 9] that many results in mathematical programming previously established for convex functions actually hold for invex functions.

Recently, Jeyakumar and Mond[3] introduced a new class of functions, namely *V*-invex functions and established sufficient optimality criteria and duality results in multiobjective static case for weak minimum solutions. In this paper we extend the concept of *V*-invexity to continuous case and use it to generalize earlier sufficient optimality and duality results for a class of multiobjective variational problems (e.g. Bector and Husain[1]).

2. PRELIMINARIES AND DEFINITIONS

Let $I = [0, T]$ be a real interval; let $f_i : I \times X_0 \times X_0 \to R$, $i = 1, 2, ..., p$ and $g : I \times X_0 \times X_0 \to R^m$ be continuously differentiable functions. X_0 denote the space of piecewise smooth functions $x : I \to R^n$, with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$u = Dx \Leftrightarrow x(t) = \alpha + \int_0^t u(s)\, ds$$

where α is a given boundary value. Thus $D = \frac{d}{dt}$ except at discontinuities. Denote the partial derivative of $f(t, x(t), \dot{x}(t))$ by

$$f_t, f_x = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right]$$

and

$$f_{\dot{x}} = \left[\frac{\partial f}{\partial \dot{x}_1}, \dots, \frac{\partial f}{\partial \dot{x}_n} \right].$$

Indexing of the vector valued function $f_i : I \times X_0 \times X_0 \to R$ have been interchanged to upper subscript for convenience in some of the proofs and statements of the theorems. For notational convenience $f(t, x(t), \dot{x}(t))$ will be written $f(t, x, \dot{x})$.

Let $F_i : X_0 \to R$ defined by

$$F_i(x) = \int_0^T f_i(t, x(t), \dot{x}(t))\, dt$$

be Frechet differentiable.

Definition 2.1 *(V-invex)* — A vector function $F = (F_1, \dots, F_p)$ is said to be V-invex if there exist differentiable vector functions $\eta(t, x, \bar{x}) : I \times X_0 \times X_0 \to R^p$ with $\eta(t, \bar{x}, \bar{x}) = 0$ and $\alpha_i : I \times X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for each $x, \bar{x} \in X_0$, and for $i = 1, 2, \dots, p$

$$F_i(x) - F_i(\bar{x}) \geq \int_0^T \{\alpha_i(t, x(t), \bar{x}(t)) f_x^i(t, \bar{x}(t), \dot{\bar{x}}(t))\, \eta(t, x(t), \bar{x}(t))$$

$$+ \frac{d}{dt} \eta_i(t, x(t), \bar{x}(t))\, \alpha_i(t, x(t), \bar{x}(t)) f_{\dot{x}}^i(t, \bar{x}(t), \dot{\bar{x}}(t))\}\, dt.$$

Definition 2.2 *(V-pseudo-invex)* — The vector function $F = (F_1, \dots, F_p)$ is said to be V-pseudo-invex if there exist functions : $\eta : I \times X_0 \times X_0 \to R^p$ with $\eta(t, \bar{x}, \bar{x}) = 0$ and $\beta_i : I \times X_0 \times X_0 \to R_+ \backslash \{0\}$ such that for each $x, \bar{x} \in X_0$,

$$\int_0^T \sum_{i=1}^p \{\eta(t, x, \bar{x}) f_x^i(t, \bar{x}, \dot{\bar{x}}) + \frac{d}{dt} \eta(t, x, \bar{x}) f_{\dot{x}}^i(t, \bar{x}, \dot{\bar{x}})\}\, dt \geq 0$$

$$\Rightarrow \int_0^T \sum_{i=1}^p \beta_i(t, x, \bar{x}) f_i(t, x, \dot{x})\, dt \geq \int_0^T \sum_{i=1}^p \beta_i(t, x, \bar{x}) f_i(t, \bar{x}, \dot{\bar{x}})\, dt;$$

or equivalently;

$$\int_0^T \sum_{i=1}^p \beta_i(t,x,\bar{x}) f_i(t,x,\dot{x})\,dt < \int_0^T \sum_{i=1}^p \beta_i(t,x,\bar{x}) f_i(t,\bar{x},\dot{\bar{x}})\,dt$$

$$\Rightarrow \int_0^T \sum_{i=1}^p \left\{ \eta(t,x,\bar{x}) f_x^i(t,\bar{x},\dot{\bar{x}}) + \frac{d}{dt}\eta(t,x,\bar{x}) f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right\} dt < 0.$$

Definition 2.3 (*V-quasi-invex*) — The vector function $F = (F_1, \ldots, F_p)$ is said to be V-quasi-invex if there exist functions $\eta : I \times X_0 \times X_0 \to R^p$ with $\eta(t,\bar{x},\bar{x}) = 0$ and $\gamma_i : I \times X_0 \times X_0 \to R_+\backslash\{0\}$ such that for each $x, \bar{x} \in X_0$,

$$\int_0^T \sum_{i=1}^p \gamma_i(t,x,\bar{x}) f_i(t,x,\dot{x})\,dt \le \int_0^T \sum_{i=1}^p \gamma_i(t,x,\bar{x}) f_i(t,\bar{x},\bar{x})\,dt$$

$$\Rightarrow \int_0^T \sum_{i=1}^p \left\{ \eta(t,x,\bar{x}) f_x^i(t,\bar{x},\dot{\bar{x}}) + \frac{d}{dt}\eta(t,x,\bar{x}) f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right\} dt \le 0;$$

or equivalently;

$$\int_0^T \sum_{i=1}^p \left\{ \eta(t,x,\bar{x}) f_x^i(t,\bar{x},\dot{\bar{x}}) + \frac{d}{dt}\eta(t,x,\bar{x}) f_{\dot{x}}^i(t,\bar{x},\dot{\bar{x}}) \right\} dt > 0$$

$$\Rightarrow \int_0^T \sum_{i=1}^p \gamma_i(t,x,\bar{x}) f_i(t,x,\dot{x})\,dt > \int_0^T \sum_{i=1}^p \gamma_i(t,x,\bar{x}) f_i(t,\bar{x},\dot{\bar{x}})\,dt.$$

It is to be noted here that, if f is independent of t, Definitions 2.1-2.3 reduce to the definitions of V-invexity, V-pseudo-invexity and V-quasi-invexity of Jeyakumar and Mond[3], respectively. It is apparent that every V-invex function is V-pseudo-invex and V-quasi-invex.

The following example shows that V-invexity does not necessarily imply invexity.

Example 2.1 — Consider

$$\min_{x_1, x_2 \in R} \left(\int_0^T \frac{x_1^2(t)}{x_2(t)}\,dt,\ \int_0^T \frac{x_2(t)}{x_1(t)}\,dt \right)$$

subject to

$$1 - x_1(t) \le 0, \quad 1 - x_2(t) \le 0.$$

Then, for

$$\alpha_1(x(t), u(t)) = \frac{u_2(t)}{x_2(t)}, \quad \alpha_2(x(t), u(t)) = \frac{u_1(t)}{x_1(t)},$$

$$\beta_i(x(t), u(t)) = 1 \text{ for } i = 1, 2, \quad \eta(x(t), u(t)) = x(t) - u(t),$$

we shall show that

$$\int_0^T \{f_i(t,x,\dot{x}) - f_i(t,u,\dot{u}) - \alpha_i(t,x,u) f_x^i(t,u,\dot{u})\,\eta(t,x,u)\}\ dt \geq 0,$$

for $i = 1, 2$.

Now,

$$\int_0^T \frac{x_1^2(t)}{x_2(t)}\,dt - \int_0^T \frac{u_1^2(t)}{u_2(t)}\,dt$$

$$-\int_0^T \frac{u_2(t)}{x_2(t)} \left(\frac{2u_1(t)}{u_2(t)},\ \frac{-u_1^2(t)}{u_2^2(t)} \right) (x_1(t)-1)\,(x_2(t)-1)\,dt$$

$$= \int_0^T \frac{x_1^2(t)}{x_2(t)}\,dt - \int_0^T 1\,dt - \int_0^T \frac{1}{x_2(t)}\,(2,-1)\,(x_1(t)-1)\,(x_2(t)-1)\,dt$$

$$= \int_0^T \frac{x_1^2(t)}{x_2(t)}\,dt - \int_0^T 1\,dt - \int_0^T \frac{1}{x_2(t)}\,(2x_1(t) - 2 - x_2(t) + 1)\,dt$$

$$= \int_0^T \frac{x_1^2(t)}{x_2(t)}\,dt - \int_0^T 1\,dt - \int_0^T \left\{ \frac{2x_1(t)}{x_2(t)} - 1 - \frac{1}{x_2(t)} \right\} dt$$

$$= \int_0^T \left\{ \frac{x_1^2(t)}{x_2(t)} - \frac{2x_1(t)}{x_2(t)} + \frac{1}{x_2(t)} \right\} dt$$

$$= \int_0^T \left\{ \frac{x_1^2(t) - 2x_1(t) + 1}{x_2(t)} \right\} dt$$

$$= \int_0^T \left\{ \frac{(x_1(t)-1)^2}{x_2(t)} \right\} dt \geq 0.$$

Thus, V-invexity does not necessarily imply invexity.

The following example shows that V-invex functions can be formed from certain nonconvex functions :

Example 2.2 — Consider the function $h : I \times X_0 \times X_0 \to R^p$

$$h(t, x(t), \dot{x}(t)) = \left(\int_0^T f_1(\phi(t, x(t), \dot{x}(t))\,dt, \ldots, \int_0^T f_p(\phi(t, x(t), \dot{x}(t))\ dt)\right),$$

where,

$$f_i : I \times X_0 \times X_0 \to R,\ i = 1, 2, \ldots, p,$$

are strongly pseudo-convex functions with real positive functions $\alpha_i(t, x(t), u(t))$,

$\phi : I \times X_0 \times X_0 \to R^n$ is surjective with $\phi'(t, u(t), \dot{u}(t))$ onto for each $u \in R^n$. Then the function h is V-invex. To see this, let $x, u \in X_0$, $v = \phi(t, x, \dot{x})$, $w = \phi(t, u, \dot{u})$. Then, by strong-pseudo-convexity;

$$\int_0^T \{f_i(\phi(t, x(t), \dot{x}(t)) - f_i(\phi(t, u(t), \dot{u}(t))\} \, dt$$

$$= \int_0^T f_i(v)\, dt - \int_0^T f_i(w)\, dt$$

$$\geq \int_0^T \alpha_i(t, v, w) f_i'(w)(v - w)\, \phi_x(t, x(t), \dot{x}(t))\cdot$$

$$+ \int_0^T \frac{d}{dt} \alpha_i(t, v, w)(v - w) f_i'(w)\, \phi_{\dot{x}}(t, x(t), \dot{x}(t)).$$

Since $\phi'(t, u(t), \dot{u}(t))$ is onto, $v - w = \phi'(t, u(t), \dot{u}(t))\, \eta(t, x(t), u(t))$ is solvable for some $\eta(t, x(t), u(t))$.

Hence

$$\int_0^T f_i(\phi(t, x(t), \dot{x}(t)) - \int_0^T f_i(\phi(t, u(t), \dot{u}(t))$$

$$\geq \int_0^T \alpha_i(t, u(t), w(t))(f_i \circ \phi)_x\, dt + \int_0^T \frac{d}{dt} \alpha_i(t, v, w)\, \eta(t, v(t), w(t))$$

$$(f_i \circ \phi)_x\, dt. \quad \square$$

Now consider the determination of a piecewise smooth extremal $x = x(t)$, $0 \leq t \leq T$, for the following problem :

Problem I (Primal = P)

$$V\text{-Minimise} \left(\int_0^T f_1(t, x, \dot{x}), \int_0^T f_2(t, x, \dot{x}), \ldots, \int_0^T f_p(t, x, \dot{x}) \right) \qquad \ldots (1)$$

subject to

$$x(0) = \alpha, \; x(T) = \beta \qquad \ldots (2)$$

and

$$g(t, x, \dot{x}) \leq 0, \quad t \in I. \qquad \ldots (3)$$

Consider also the determination of $m + n$ dimensional extremal $(u, y) = (u(t), y(t))$, $t \in I$, for the following maximisation problem :

Problem II (Dual = D)

$$V\text{-maximise}\left(\int_0^T f_1(t,u,\dot u),\ \int_0^T f_2(t,u,\dot u),\ \dots,\ \int_0^T f_p(t,u,\dot u)\right) \qquad \dots (4)$$

subject to

$$u(0) = \alpha,\ u(T) = \beta \qquad \dots (5)$$

$$\sum_{i=1}^{p} \tau_i f_u^i(t,u,\dot u) + \sum_{i=1}^{m} y_j g_u^j(t,u,\dot u)$$

$$= \frac{d}{dt}\left\{\sum_{i=1}^{p} \tau_i f_{\dot u}^i(t,u,\dot u) + \sum_{j=1}^{m} y_j(t)\ g_{\dot u}(t,u,\dot u)\right\} \qquad \dots (6)$$

$$\int_0^T y_j(t) g_j(t,u,\dot u)\,dt \geq 0,\ j = 1, \dots, m \qquad \dots (7)$$

$$y(t) \geq 0,\ t \in I,\ \tau e = 1,\ \tau \geq 0 \qquad \dots (8)$$

where

$$e = (1, 1, \dots, 1) \in R^p.$$

Recall from Jeyakumar and Mond[3] that a point x_0 is said to be a (global) weak minimum if there exists no other feasible point x for which

$$\int_0^T f_i(t, x_0, \dot x_0)\,dt > \int_0^T f_i(t,x,\dot x)\,dt, \text{ for } i = 1, 2, \dots, p.$$

3. Sufficient Optimality Criteria and Duality Theorems

In this section, we present sufficient optimality criteria of the Kuhn-Tucker type for the problem (P).

Theorem 3.1 *(Sufficient optimality conditions)* — Let x^* be a feasible solution of the problem (P) and assume that

$$\left(\int_0^T \tau_1 f_1(t,\cdot,\cdot)\,dt,\ \dots,\ \int_0^T \tau_p f_p(t,\cdot,\cdot)\,dt\right)$$

is V-pseudo-invex with respect to η and

$$\left(\int_0^T y_1 g_1(t,\cdot,\cdot)\,dt,\ \dots,\ \int_0^T y_m g_m(t,\cdot,\cdot)\,dt\right)$$

is V-quasi-invex with respect to η. If there exists a piecewise smooth $y^* : I \to R^m$

such that $(x^*(t), y^*(t))$ satisfies the conditions :

$$\sum_{i=1}^{p} \tau_i f_x^i(t, x^*, \dot{x}^*) + \sum_{j=1}^{m} y_j^*(t)\, g_x^j(t, x^*, \dot{x}^*)$$

$$= \frac{d}{dt}\left\{ \sum_{i=1}^{p} \tau_i f_{\dot{x}}^i(t, x^*, \dot{x}^*) + \sum_{j=1}^{m} y_j^*(t)\, g_{\dot{x}}^j(t, x^*, \dot{x}^*) \right\} \quad \text{... (9)}$$

$$y_j^*(t)\, g_j(t, x^*, \dot{x}^*) = 0, \; t \in I \quad \text{... (10)}$$

$$\tau \in R^p, \; \tau \neq 0, \; \tau \geq 0, \; y^* \in R^m, \; y^*(t) \geq 0, \; t \in I, \quad \text{... (11)}$$

then x^* is a global weak minimum for (P).

PROOF : Suppose that x^* is not a global weak minimum point. Then there exists feasible $x_0 \in X_0$ such that

$$\int_0^T f_i(t, x_0(t), \dot{x}_0(t))\, dt < \int_0^T f_i(t, u(t), \dot{u}_0(t))\, dt,$$

$$i = 1, 2, \ldots, p.$$

So

$$\int_0^T \sum_{i=1}^{p} \beta_i(t, x_0(t), u(t))\, \tau_i\, f_i(t, x_0(t), \dot{x}_0(t))\, dt$$

$$< \int_0^T \sum_{i=1}^{p} \beta_i(t, x_0(t), u(t))\, \tau_i\, f_i(t, u(t), \dot{u}(t))\, dt.$$

Now, by the V-pseudo-invexity condition, we get

$$\int_0^T \Big[\sum_{i=1}^{p} \tau_i \Big\{ \eta(t, x_0(t), u(t))\, f_x^i(t, u(t), \dot{u}(t))$$

$$+ \left(\frac{d}{dt} \eta(t, x_0(t), u(t)) \right) f_{\dot{x}}^i(t, u(t), \dot{u}(t)) \Big\} \Big]\, dt < 0. \quad \text{... (12)}$$

From (9) we have

$$\int_0^T \eta(t, x_0(t), u(t)) \Big[\sum_{i=1}^{p} \tau_i\, f_x^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*\, g_x^j(t, u(t), \dot{u}(t)) \Big]\, dt$$

$$= \int_0^T \eta(t, x_0(t), u(t)) \; \frac{d}{dt} \Big[\sum_{i=1}^{p} \tau_i \, f_{\dot{x}}^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*(t) \, g_{\dot{x}}^j(t, u(t), \dot{u}(t)) \Big] dt$$

$$= \eta(t, x_0(t), u(t)) \Big[\sum_{i=1}^{p} \tau_i \, f_{\dot{x}}^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*(t) \, g_{\dot{x}}^j(t, u(t), \dot{u}(t)) \Big|_{t=0}^{t=T}$$

$$- \int_0^T \left(\frac{d}{dt} \eta(t, x_0(t), u(t)) \right) \Big[\sum_{i=1}^{p} \tau_i \, f_{\dot{x}}^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*(t) \, g_{\dot{x}}^j(t, u(t), \dot{u}(t)) \Big] \text{ (integration by parts).}$$

Thus

$$\int_0^T \eta(t, x_0(t), u(t)) \Big\{ \sum_{i=1}^{p} \tau_i \, f_x^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*(t) \, g_x^j(t, u(t), \dot{u}(t)) \Big\} \, dt$$

$$+ \int_0^T \left(\frac{d}{dt} \eta(t, x_0(t), u(t)) \right) \Big\{ \sum_{i=1}^{p} \tau_i \, f_{\dot{x}}^i(t, u(t), \dot{u}(t))$$

$$+ \sum_{j=1}^{m} y_j^*(t) \, g_{\dot{x}}^j(t, u(t), \dot{u}(t)) \Big\} \, dt = 0 \qquad \dots (13)$$

(since $\eta(t, u, u) = 0$)

From (13), we have

$$\int_0^T \sum_{j=1}^{m} \Big\{ \eta(t, x_0(t), u(t)) \, y_j^*(t) \, g_x^j(t, u(t), \dot{u}(t))$$

$$+ \left(\frac{d}{dt} \eta(t, x_0(t), u(t)) \right) y_j^*(t) \, g_{\dot{x}}^j(t, u(t), \dot{u}(t)) \Big\} \, dt$$

$$= -\int_0^T \sum_{i=1}^{p} \{\eta(t, x_0(t), u(t))\, \tau_i f_x^i(t, u(t), \dot{u}(t))$$

$$+ \left(\frac{d}{dt}\eta(t, x_0(t), u(t))\right)\tau_i f_{\dot{x}}^i(t, u(t), \dot{u}(t))\}\ dt. \qquad \text{... (14)}$$

From (14), and (12), we have

$$\int_0^T \sum_{j=1}^{m} \{\eta(t, x_0(t), u(t))\, y_j^*(t)\, g_x^j(t, u(t), \dot{u}(t))$$

$$+ \left(\frac{d}{dt}\eta(t, x_0(t), u(t))\right) y_j^*(t)\, g_{\dot{x}}^j(t, u(t), \dot{u}(t))\}\ dt > 0. \qquad \text{... (15)}$$

Now (15) in view of V-quasi-invexity of $\left(\int_0^T y_1 g_1(t, \cdot, \cdot), \ldots, \int_0^T y_m g_m(t, \cdot, \cdot)\right)$ yields

$$\int_0^T \sum_{j=1}^{m} \gamma_j(t, x_0(t), u(t))\, g_j(t, x_0(t), \dot{x}_0(t))\, dt$$

$$> \int_0^t \sum_{j=1}^{m} \gamma_j(t, x_0(t), u(t))\, g_j(t, u(t), \dot{u}(t))\, dt.$$

This is a contradiction, since

$$\gamma_j(t, x_0(t), u(t))\, g_j(t, x_0(t), \dot{x}_0(t)) \leq 0,$$

$$\gamma_j(t, x_0(t), u(t))\, g_j(t, u(t), \dot{u}(t)) = 0$$

and

$$\gamma_j(t, x_0(t), u(t)) > 0 \text{ for } j = 1, 2, \ldots, m.$$

We now establish duality results between (P) and (D). Note that the problem (D) is known as Mond-Weir type dual.

Theorem 3.2 *(Weak duality)* — Let x be feasible for (P) and let (ξ, τ, λ) be feasible for (D). If $(\int_0^T \tau_1 f_1(t, \cdot, \cdot), \ldots, \int_0^T \tau_p f_p(t, \cdot, \cdot))$ is V-pseudo-invex and

$$(\int_0^T \lambda_1 g_1(t, \cdot, \cdot), \ldots, \int_0^T \lambda_m g_m(t, \cdot, \cdot))$$

is V-quasi-invex with respect to the same η, then

$$\left(\int_0^T f_1(t, x(t), \dot{x}(t))\, dt, \ldots, \int_0^T f_p(t, x(t), \dot{x}(t))\, dt\right)$$

$$-\left(\int_0^T f_1(t, \xi(t), \dot{\xi}(t))\, dt, \ldots, \left(\int_0^T f_p(t, \xi(t), \dot{\xi}(t))\, dt\right)^T \notin -\operatorname{int} R_+^p.$$

PROOF : From the feasibility conditions,

$$\int_0^T \lambda_j g_j(t, x(t), \dot{x}(t))\, dt \leq \int_0^T \lambda_j g_j(t, \xi(t), \dot{\xi}(t))\, dt),$$

for each $j = 1, 2, \ldots, m$.
Since $\gamma_j(t, x(t), \xi(t))$ is positive,

$$\int_0^T \sum_{j=1}^m \gamma_j(t, x(t), \xi(t))\, \lambda_j g_j(t, x(t), \dot{x}(t))\, dt$$

$$\leq \int_0^T \sum_{j=1}^m \gamma_j(t, x(t), \xi(t))\, \lambda_j g_j(t, \xi(t), \dot{\xi}(t))\, dt. \qquad \ldots (16)$$

Hence

$$\int_0^T \sum_{j=1}^m \Big\{ \eta(t, x(t), \xi(t))\, \lambda_j g^j(t, \xi(t), \dot{\xi}(t))$$

$$+ \left(\frac{d}{dt} \eta(t, x(t), \xi(t))\right) \lambda_j g^j(t, \xi(t), \dot{\xi}(t)) \Big\}\, dt \leq 0. \qquad \ldots (17)$$

The constraint (6), as earlier, is equivalent to

$$-\int_0^T \sum_{j=1}^m \Big\{ \eta(t, x(t), \xi(t))\, \lambda_j\, g^j_\xi(t, \xi(t), \dot{\xi}(t))$$

$$+ \left(\frac{d}{dt} \eta(t, x(t), \xi(t))\right) \lambda_j\, g^j_{\dot{\xi}}(t, \xi(t), \dot{\xi}(t)) \Big\}$$

$$= \int_0^T \sum_{j=1}^p \Big\{ \eta(t, x(t), \xi(t))\, \tau_i\, f^i_\xi(t, \xi(t), \dot{\xi}(t))$$

$$+ \left(\frac{d}{dt} \eta(t, x(t), \xi(t))\right) \tau_i\, f^i_{\dot{\xi}}(t, \xi(t), \dot{\xi}(t)) \Big\}\, dt. \qquad \ldots (18)$$

Using (17), (18) yields

$$\int_0^T \sum_{i=1}^p \Big\{ \eta(t, x(t), \xi(t)) \tau_i f_\xi^i(t, \xi(t), \dot{\xi}(t))$$

$$+ \left(\frac{d}{dt}\eta(t, x(t), \xi(t))\right) \tau_i f_{\dot{\xi}}^i(t, \xi(t), \dot{\xi}(t)) \Big\}\, dt \geq 0. \qquad \dots (19)$$

The conclusion now follows from the V-pseudo-invexity condition on

$$\left(\int_0^T \tau_1 f_1(t, \cdot, \cdot), \dots, \int_0^T \tau_p f_p(t, \cdot, \cdot)\right)$$

since $\tau_e = 1$, and $\beta_i(t, x(t), \xi(t)) > 0$.

Theorem 3.3 *(Strong duality)* — Assume that ξ is a weak minimum for (P) and that a suitable constraint qualification is satisfied at ξ. Then there exist (τ, λ) such that (ξ, τ, λ) is feasible for (D) and the objective functions of (P) and (D) are equal at these points. If also, for all feasible (ξ, τ, λ), V-pseudo-invexity and V-quasi-invexity conditions as in Theorem 3.2 hold, then (ξ, τ, λ) is weak maximum for (D).

PROOF : Since ξ is a weak minimum for (P) and constraint qualification is satisfied at ξ, from the Lagrangian conditions (Theorem 3.1), there exists (τ, λ) such that (ξ, τ, λ) is feasible for (D). Clearly the values of (P) and (D) are equal at ξ, since the objective functions for both problems are the same. By the generalised V-invexity hypotheses, weak duality holds; hence if (ξ, τ, λ) is not a weak optimum for (D), there must exist (a, τ^*, λ^*) feasible for (D), $u \neq \xi$, such that

$$\left(\int_0^T f_1(t, u(t), \dot{u}(t)), \dots, \int_0^T f_p(t, u(t), \dot{u}(t))\right)^T$$

$$- \left(\int_0^T f_1(t, \xi(t), \dot{\xi}(t)), \dots, \int_0^T f_p(t, \xi(t), \dot{\xi}(t))\right)^T \in \operatorname{int} R_+^p.$$

contradicting weak duality.

4. NONCONVEX MULTI-OBJECTIVE FRACTIONAL VARIATIONAL PROBLEMS

In this section we apply the results of the previous section to study fractional nonconvex multi-objective variational problem.

Consider the fractional variational problem

$$\text{(F) } v\text{-minimise} \left(\frac{\int_0^T p_1(t, x(t), \dot{x}(t))\, dt}{\int_0^T q_1(t, x(t), \dot{x}(t))\, dt}, \dots, \frac{\int_0^T p_r(t, x(t), \dot{x}(t))\, dt}{\int_0^T p_r(t, x(t), \dot{x}(t))\, dt} \right) \qquad \dots (20)$$

subject to $x(0) = \alpha$, $x(T) = \beta$ $\qquad \dots (21)$

and

$$g(t,\ x(t),\ \dot{x}(t)) \leq 0, \qquad \text{... (22)}$$

where $p_i : I \times X_0 \times X_0 \to R$, $q_i : I \times X_0 \times X_0 \to R$ and $g : I \times X_0 \times X_0 \to R^m$. We assume that

$$\int_0^T p_i\,(t, x(t), \dot{x}(t))\, dt \geq 0,\ i = 1,\ ...,\ r,$$

for each x on the feasible set

$$\Delta = \{x \in X_0 : g(t,\ x(t),\ \dot{x}(t) \leq 0\}, \qquad \int_0^T q_i\,(t, x(t), \dot{x}(t))\, dt > 0$$

for each $x \in \Delta$, $i = 1, 2, ..., r$. The problem (F) is said to be a V-invex fractional variational problem if the functions p, q and g satisfy

$$x,\ a \in \Delta \Rightarrow \int_0^T p_i\,(t, x(t), \dot{x}(t))\, dt - \int_0^T p_i\,(t, u(t), \dot{u}(t))\, dt$$

$$\geq \int_0^T \{\gamma_i\,(t, x(t), u(t))\ \ p_x^i\,(t, u(t), \dot{u}(t))\, \eta\,(t, x(t), u(t))$$

$$+ \left(\frac{d}{dt}\eta\,(t, x(t), u(t))\, \gamma_i\,(t, x(t), u(t))\right) p_{\dot{x}}^i\,(t, u(t), \dot{u}(t))\}\ dt,$$

q_i satisfies reverse inequality of the above and g satisfies same as above, with γ, η and β as earlier.

The problem (F) is said to be convex-concave fractional variational problem if p_i is convex, q_i is concave and g is convex. If p, q and g are independent of t, then (F) reduces to (FI) of Jeyakumar and Mond[3].

The equivalent parametric form of the problem (F) for $v \in R_+$ is :

$$\text{(EF)}\ V\text{-minimise} \left(\int_0^T \{p_1(t, x(t), \dot{x}(t)) - v q_1\,(t, x(t), \dot{x}(t))\}\ dt,\right.$$

$$\left. ..., \int_0^T \{p_r(t, x(t), \dot{x}(t)) - v q_r\,(t, x(t), \dot{x}(t))\}\ dt \right) \qquad \text{... (23)}$$

subject to $\qquad x(0) = \alpha,\ x(T) = \beta \qquad$ ... (24)

and

$$g(t, x(t), \dot{x}(t)) \leq 0. \qquad \text{... (25)}$$

Theorem 4.1 *(Sufficient optimality conditions)* — Consider the V-invex problem (23)-(25). Let $u \in \Delta$. Assume that there exist (τ, λ) such that $\tau \geq 0$, $\tau \neq 0$, $\lambda \geq 0$,

$$\sum_{i=1}^{r} \tau_i \{p_x^i(t, u(t), \dot{u}(t)) - \nu\, q_x^i(t, u(t), \dot{u}(t))\} + \sum_{j=1}^{m} \tau_j g_x^j(t, u(t), \dot{u}(t))$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{r} \tau_i \left\{p_{\dot{x}}^i(t, u(t), \dot{u}(t)) - \nu\, q_{\dot{x}}^i(t, u(t), \dot{u}(t))\right\}\right],$$

$$\int_0^T \lambda_j g_j(t, u(t), \dot{u}(t))\, dt = 0 \text{ for } j = 1, \ldots, m, \ \tau_e = 1, \ \tau \geq 0.$$

Then, u is a global weak minimum for (F).

Proof will follow the lines of the proof of Theorem 3.1.

ACKNOWLEDGEMENT

The authors are thankful to the referees for useful comments which led to the present improved version of the paper as it stands.

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