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THE OPTIMAL ARRANGEMENT OF COMMUNICATION CENTRES LINKED BY A TRUNK

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This article discusses the formation of linear communication systems, with communication centres in predetermined communication areas, by a single trunk linked in linear order. A mathematical model is developed for studying the effects of various arrangement of communication centres upon general communication flow.

1. INTRODUCTION

Virtually all telecommunication networks are arranged in a hierarchical structure to achieve efficient loading of transmission and switching facilities[2]. Calls and messages of local telecommunication are routed from low-level terminal devices through increasingly higher levels of aggregation; once the highest level (it is called a communication centre of this area) of the network is reached, the process is reversed to the distant terminal device of the same area or another areas. This article will study methods of linking area communication centres in order to adopt linear communication.

Generally, linear communication may be described as follows : Assume that there are communication demand areas $R_1, R_2, ..., R_n$, in which the communication demand at R_i is N_i, and satisfies the inequality

$$N_1 > N_2 > \ldots > N_n. \qquad \ldots (1.1)$$

It is known that each area R_i has a communication exchange centre C_i handling communications inside and outside of this area and being in charge of extension work. In order to communicate with people in different areas, arranging communication centres in all areas into a linear sequence $C_{g(1)}, C_{g(2)}, ..., C_{g(n)}$, will be considered where g is a one to one correspondence function from $\{1, 2, ..., n\}$ mapping to itself; then a pair of proximate area communication centres $C_{g(k)}$ and $C_{g(k+1)}$ are linked with a single circuit to form a trunk. We call this circuit linking method a linear communication system (see Fig. 1).

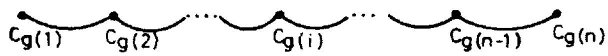

FIG. 1. Linear communication system.

Two kinds of problems arise in designing linear communication systems : (1) how to divide each communication area R_i (communication bridge installing position problem), and (2) how to link each divided communication area in linear order. Boffey and Karkazis[1] and Martin[3] have already looked into the first mentioned problem; but the second problem has not been considered further in any literature so far. This article will try to turn the second problem into a mathematical model. Certain concrete results are achieved through careful discussion as reference criteria for linear communication system design.

2. THE MODEL

Let $S = \{1, 2, ..., n\}$, $n \geq 3$, and S_n be the set of possible arrangements for linear communication of n communication centres $C_1, C_2, ..., C_n$, namely, $S_n = \{g \mid g$ is one to one correspondence function of $S \to S\}$. Each element g in S_n corresponds exactly to one linear arrangement of C_i's (as in Fig. 1), and obviously S_n contains $n!$ different elements.

Because differences in communication demand for different people and occasions can hardly be taken into account in a general model, this paper assumes that each person's potential communication demand is the same, and area communication related numbers will be used as the communication flow index.

In arrangement method g,

$$R_{g(t)} \text{ within area communication flow is } [N_{g(t)}]^2. \qquad \text{... (2.1)}$$

The communication flow between $R_{g(t)}$ and $R_{g(1)}, R_{g(2)}, ..., R_{g(t-1)}$,

$$R_{g(t+1)}, ..., R_{g(n)} \text{ is } 2N_{g(t)} \sum_{r \neq t} N_{g(r)}. \qquad \text{... (2.2)}$$

Because the communication between area $R_{g(1)} \bigcup R_{g(2)} \bigcup .... \bigcup R_{g(t-1)}$ and area $R_{g(t+1)} \bigcup ... \bigcup R_{g(n)}$ must be reached through the transmitting centre of $R_{g(t)}$, using $H_g(R_{g(t)})$ to represent the communication flow between area $R_{g(1)} \bigcup R_{g(2)} \bigcup ... \bigcup R_{g(t-1)}$ and area $R_{g(t+1)} \bigcup ... \bigcup R_{g(n)}$ leads to

$$H_g(R_{g(t)}) = 2 \sum_{i<t} N_{g(i)} \sum_{j>t} N_{g(j)}. \qquad \text{... (2.3)}$$

By (2.1), (2.2) and (2.3), the communication flow to be handled by the communication centre $C_{g(t)}$ is

$$F_g(R_{g(t)}) = D_g(R_{g(t)}) + H_g(R_{g(t)}) \qquad \text{... (2.4)}$$

where $$D_g(R_{g(t)}) = [N_{g(t)}]^2 + 2N_{g(t)} \sum_{r \neq t} N_{g(r)}$$

$$= (N_{g(t)})^2 + 2N_{g(t)} \left(\sum_{r=1}^{t-1} N_{g(r)} + \sum_{r=t+1}^{n} N_{g(r)} \right) \quad \ldots (2.5)$$

Since the communication flow of the whole linear communication system is the sum of communication flows handled by each area communication centre, we use $F(g)$ to represent the communication flow of the whole system. Then

$$F(g) = \sum_{t=1}^{n} F_g(R_{g(t)}) \; ; \quad \text{by (2.4) and (2.5)}$$

$$= \sum_{t=1}^{n} D_g(R_{g(t)}) + \sum_{t=1}^{n} H_g(R_{g(t)}) \quad \ldots (2.6)$$

where $$\sum_{t=1}^{n} D_g(R_{g(t)})$$

$$= \sum_{t=1}^{n} \left[(N_{g(t)})^2 + 2N_{g(t)} \left(\sum_{r=1}^{t-1} N_{g(r)} + \sum_{r=t+1}^{n} N_{g(r)} \right) \right]$$

$$= \sum_{t=1}^{n} (N_{g(t)})^2 + 2 \sum_{t=1}^{n} N_{g(t)} \left(\sum_{r=1}^{n} N_{g(r)} - N_{g(t)} \right)$$

$$= 2\left(\sum_{t=1}^{n} N_{g(t)} \right)^2 - \sum_{t=1}^{n} (N_{g(t)})^2$$

$$= 2\left(\sum_{i=1}^{n} N_i \right)^2 - \sum_{i=1}^{n} N_i^2 \quad \ldots (2.7)$$

$$\sum_{t=1}^{n} H_g(R_{g(t)})$$

$$= 2 \sum_{t=1}^{n} \left(\sum_{\substack{i<t \\ j>t}} N_{g(i)} N_{g(j)} \right)$$

$$= 2\left(\sum_{1 \le i < j \le n} N_{g(i)} N_{g(j)} \right) \left(\sum_{\substack{t \\ i<t<j}} 1 \right)$$

$$= 2 \sum_{1 \le i < j \le n} (j - i - 1) N_{g(i)} N_{g(j)}. \quad \ldots (2.8)$$

Generally speaking, the increase in communication flow will make the loss caused by communication delay increase. Our paper will only discuss the extent of the effect produced by different arrangements upon communication flow, and the factors which affect communication flow of the whole linear communication system; therefore, the aim of the model is to look for the optimal arrangement $g \in S_n$ to minimize $F(g)$. Namely,

$$\min_{g \in S_n} F(g) . \qquad \text{... (2.9)}$$

From (2.7), we note that the first term of (2.6) is independent of g. So, from (2.8) if follows that problem (2.9) and the following problem have the same optimal solution :

$$\min_{g \in S_n} G(g) \qquad \text{... (2.10)}$$

where $$G(g) = \sum_{i<j} (j-i-1) \; N_{g(i)} N_{g(j)}.$$

3. The Optimal Solution

Lemma 1 — Given a feasible solution $g \in S_n$ and two positive integers p and q with $p + q \leq n$. Let $g_{pq} \in S_n$, be defined by

$$g_{pq}(i) = \begin{cases} g(p+q) & \text{if } i = p \\ g(p) & \text{if } i = p+q \\ g(i) & \text{if } i \neq p \text{ and } i \neq p+q. \end{cases} \qquad \text{... (3.1)}$$

That is, swap positions of $C_{g(p)}$ and $C_{g(p+q)}$

then $$G(g_{pq}) - G(g) = [N_{g(p+q)} - N_{g(p)}]$$

$$\left[q \left(\sum_{p+q<i} N_{g(i)} - \sum_{i<p} N_{g(i)} \right) + \sum_{p<i<p+q} (2i-2p-q) N_{g(i)} \right].$$

Proof : $G(g_{pq}) - G(g)$

$$= \sum_{i<j} (j-i-1) \, [N_{g_{pq}(i)} N_{g_{pq}(j)} - N_{g(i)} N_{g(j)}] \; ; \text{ by (3.1)}$$

$$= \sum_{i<p} (p-i-1) \, [N_{g(i)} N_{g(p+q)} - N_{g(i)} N_{g(p)}] + \sum_{\substack{i<p+q \\ i \neq p}} (p+q-i-1)$$

$$[N_{g(i)} N_{g(p)} - N_{g(i)} N_{g(p+q)}] + \sum_{\substack{p<j \\ j \neq p+q}} (j-p-1) \, [N_{g(p+q)} N_{g(j)} - N_{g(p)} N_{g(j)}]$$

$$+ \sum_{p+q<j} (j-p-q-1) \, [N_{g(p)} N_{g(j)} - N_{g(p+q)} N_{g(j)}]$$

$$= N_{g(p+q)} \left[\sum_{i<p} (p-i-1) N_{g(i)} - \sum_{\substack{i<p+q \\ i \neq p}} (p+q-i-1) N_{g(i)} \right.$$

$$\left. + \sum_{\substack{p<j \\ j \neq p+q}} (j-p-1) N_{g(j)} - \sum_{p+q<j} (j-p-q-1) N_{g(j)} \right]$$

$$- N_{g(p)} \left[\sum_{i<p} (p-i-1) N_{g(i)} - \sum_{\substack{i<p+q \\ i \neq p}} (p+q-i-1) N_{g(i)} \right.$$

$$\left. + \sum_{\substack{p<j \\ j \neq p+q}} (j-p-1) N_{g(j)} - \sum_{p+q<j} (j-p-q-1) N_{g(j)} \right]$$

$$= [N_{g(p+q)} - N_{g(p)}] \left[\left(\sum_{i<p} (p-i-1) N_{g(i)} - \sum_{\substack{i<p+q \\ i \neq p}} (p+q-i-1) N_{g(i)} \right) \right.$$

$$\left. + \left(\sum_{\substack{p<j \\ j \neq p+q}} (j-p-1) N_{g(j)} - \sum_{p+q<j} (j-p-q-1) N_{g(j)} \right) \right]$$

$$= [N_{g(p+q)} - N_{g(p)}] \left[\sum_{i<p} (-q) N_{g(i)} - \sum_{p<i<p+q} (p+q-i-1) N_{g(i)} \right.$$

$$\left. + \sum_{p<j<p+q} (j-p-1) N_{g(j)} + \sum_{p+q<j} (q) N_{g(j)} \right]$$

$$= [N_{g(p+q)} - N_{g(p)}]$$

$$\left[q \left(\sum_{p+q<i} N_{g(i)} - \sum_{i<p} N_{g(i)} \right) + \sum_{p<i<p+q} (2i-2p-q) N_{g(i)} \right]$$

Lemma 2 — Suppose that $g^* \in S_n$ is an optimal solution of problem (2.10), and assume $\bar{k}$ is the largest integer k to satisfy the inequality

$$\sum_{i<k-1} N_{g^*(i)} \leq \sum_{k<i} N_{g^*(i)} \qquad \text{... (3.2)}$$

then

(1) $2 \leq \bar{k} \leq n-1$

(2) $N_{g^*(1)} \leq N_{g^*(2)} \leq \ldots \leq N_{g^*(\bar{k}-1)} \leq N_{g^*(\bar{k})}$ and

$N_{g^*(\bar{k})} \geq N_{g^*(\bar{k}-1)} \geq \ldots \geq N_{g^*(n-1)} \geq N_{g^*(n)}$

(3) $g^*(\bar{k}) = 1.$

PROOF : (1) Since $\sum_{i<1} N_{g^*(i)} = \sum_{n<i} N_{g^*(i)} = 0$, inequality (3.2) holds for $k = 2$, but it does not hold for $k = n$.

(2) Since $\bar{k}$ is the largest integer to satisfy (3.2). Then

$$\left.\begin{aligned} &\sum_{i<k-1} N_{g^*(i)} \le \sum_{k<i} N_{g^*(i)} \text{ for } k = 2, 3, \ldots, \bar{k} \\ &\sum_{i<k-1} N_{g^*(i)} > \sum_{k<i} N_{g^*(i)} \text{ for } k = \bar{k}+1, \bar{k}+2, \ldots, n. \end{aligned}\right] \quad \ldots (3.3)$$

Using Lemma 1, letting $p = k - 1$ and $q = 1$ yields

$$\begin{aligned} 0 &\le G(g^*_{k-1,1}) - G(g^*) \\ &= (N_{g^*(k)} - N_{g^*(k-1)}) \left(\sum_{k<i} N_{g^*(i)} - \sum_{i<k-1} N_{g^*(i)} \right) \quad \ldots (3.4) \\ &\qquad \text{for } k = 2, 3, \ldots, n. \end{aligned}$$

Together with (3.3) and (3.4) leads to

$$\begin{cases} N_{g^*(k-1)} - N_{g^*(k)} \le 0 & \text{for } k = 2, 3, \ldots, k \\ N_{g^*(k-1)} - N_{g^*(k)} \ge 0 & \text{for } k = \bar{k}+1, \bar{k}+2, \ldots, n. \end{cases}$$

The desired results have been established.

(3) Using the results of (2), we have

$$N_{g^*(\bar{k})} = \max_{1 \le k \le n} \left\{ N_{g^*(k)} \right\} = \max_{1 \le i \le n} \left\{ N_i \right\} = N_1$$

and therefore $g(\bar{k}) = 1$.

For any $g \in S_n$, if function $\bar{g}$ is defined by

$$\bar{g}(i) = g(n - i + 1) \quad \text{for } i = 1, 2, \ldots, n \quad \ldots (3.5)$$

then the corresponding linear connection method of C_i is shown below (see Fig. 2).

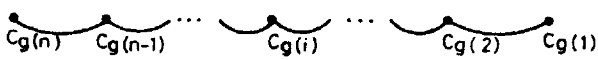

FIG. 2. Linear communication system.

In Fig. 1, if we exchange the symmetrical positions on both sides of the centre on a trunk, it will turn into Fig. 2. This implies $F(g) = F(\bar{g})$. By the definition of (3.5), $g^{-1}(2) \le g^{-1}(1)$ or $\bar{g}^{-1}(2) \le \bar{g}^{-1}(1)$ at least will hold. To generalize, all feasible solutions can be limited to the following set for consideration. $A_n = \{g \in S_n \mid g^{-1}(2) \le g^{-1}(1)\}$. Therefore the problem (2.10) can be written as follows :

$$\min_{g \in A_n} \sum_{i<j} (j - i - 1) N_{g(i)} N_{g(j)}. \quad \ldots (3.6)$$

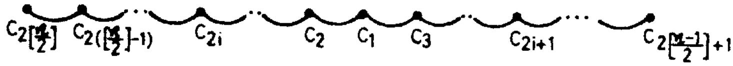

FIG. 3. The optimal linear communication system.

Theorem — If $g^* \in A_n$ is the optimal solution for problem (3.6), then the corresponding connection method is given in Fig. 3.

That is

$$g^*(i) = \begin{cases} 2([n/2] - i + 1) & \text{for } 1 \le i \le [n/2] \\ 2(i - [n/2]) - 1 & \text{for } [n/2] < i \le n\,. \end{cases}$$

PROOF : Assume that $\bar{k}$ is the largest integer to satisfy the inequality

$$\sum_{i < \bar{k} - 1} N_{g^*(i)} \le \sum_{\bar{k} < i} N_{g^*(i)}.$$

Step 1 : Using Lemma 2 and the definition on A_n, we have

$$C_{g^*(\bar{k})} = C_1,\ C_{g^*(\bar{k}-1)} = C_2$$

and $$C_{g^*(\bar{k}+1)} = C_3 \text{ or } C_{g^*(\bar{k}-2)} = C_3\,. \qquad \ldots (3.7)$$

We claim that $C_{g^*(k+1)} = C_3$ (i.e. the latter case in (3.7) can not happen), as shown below :

Suppose that $C_{g^*(\bar{k}-2)} = C_3$, the optimal solution $g^* \in A_n$ is as shown in Fig. 4.

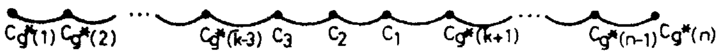

FIG. 4. Linear communication system.

Property (3.2) and Fig. 4 yields that

$$\sum_{\bar{k}+1 < i} N_{g^*(i)} + N_{g^*(\bar{k}+1)} \ge \sum_{i < \bar{k} - 2} N_{g^*(i)} + N_{g^*(\bar{k}-2)};$$

by supposition that $g^*(\bar{k} - 2) = 3$ and (1.1)

$$> \sum_{i < \bar{k} - 2} N_{g^*(i)} + N_{g^*(\bar{k}+1)}$$

and hence $$\sum_{\bar{k}+1 < i} N_{g^*(i)} > \sum_{i < \bar{k} - 2} N_{g^*(i)}. \qquad \ldots (3.8)$$

Using Lemma 1, with $p = \bar{k} - 2$ and $q = 3$ yields

$$G(g^*_{\bar{k}-2,3}) - G(g^*) = [N_{g^*_{(\bar{k}+1)}} - N_{g^*_{(\bar{k}-2)}}]$$

$$\left[3\left(\sum_{\bar{k}+1<i} N_{g^*(i)} - \sum_{i<\bar{k}-2} N_{g^*(i)}\right) + \sum_{\bar{k}-2<i<\bar{k}+1} (2i - 2\bar{k} + 1) N_{g^*(i)}\right];$$

by Fig. 4

$$= [N_{g^*(\bar{k}+1)} - N_3]\left[3\left(\sum_{\bar{k}+1<i} N_{g^*(i)} - \sum_{i<\bar{k}-2} N_{g^*(i)}\right) + (N_1 - N_2)\right];$$

by (3.8) and (1.1)

$$< 0.$$

This contradicts the assumption that g^* is an optimal solution of problem (3.6). Hence $C_{g^*(\bar{k}+1)} = C_3$ as asserted.

Step 2 : If $n \geq 4$ and $C_{g^*(\bar{k}+1)} = C_3$ then the property of Lemma 2, gives

$$C_{g^*(\bar{k}-2)} = C_4 \text{ or } C_{g^*(\bar{k}+2)} = C_4. \qquad \text{... (3.9)}$$

We claim that $C_{g^*(\bar{k}-2)} = C_4$ [i.e. the latter case in (3.9) can not happen] as shown below :

Suppose that $C_{g^*(\bar{k}+2)} = C_4$, then the optimal solution $g^* \in A_n$ is as shown in Fig. 5.

FIG. 5. Linear communication system.

If $p = \bar{k} - 1$ and $q = 2$ Lemma 1 yields

$$0 \leq G(g^*_{\bar{k}-1,2}) - G(g^*)$$

$$= [N_{g^*(\bar{k}+1)} - N_{g^*(\bar{k}-1)}]\left[2\left(\sum_{\bar{k}+1<i} N_{g^*(i)} - \sum_{i<\bar{k}-1} N_{g^*(i)}\right)\right];$$

by Figure 5

$$= [N_3 - N_2]\left[2\left(\sum_{\bar{k}+1<i} N_{g^*(i)} - \sum_{i<\bar{k}-1} N_{g^*(i)}\right)\right]. \qquad \text{... (3.10)}$$

By (3.10) and (1.1), we have

$$\sum_{i<\bar{k}-1} N_{g^*(i)} \geq \sum_{\bar{k}+1<i} N_{g^*(i)}. \qquad \text{... (3.11)}$$

This implies $\bar{k} > 2$ and hence $n > 4$. By (3.11),

$$\sum_{i<\bar{k}-2} N_{g^*(i)} + N_{g^*(\bar{k}-2)} \geq \sum_{\bar{k}+2<i} N_{g^*(i)} + N_{g^*(\bar{k}+2)}$$

$$> \sum_{\bar{k}+2<i} N_{g^*(i)} + N_{g^*(\bar{k}-2)} \text{ if } g^*(\bar{k}+2) = 4 \text{ (by (1.1))}$$

and hence $\sum_{i<\bar{k}-2} N_{g^*(i)} > \sum_{\bar{k}+2<i} N_{g^*(i)}$. ... (3.12)

Using Lemma 1, let $p = \bar{k} - 2$ and $q = 4$ and yield that

$$G(g^*_{\bar{k}-2,4}) - G(g^*)$$

$$= [N_{g^*(\bar{k}+2)} - N_{g^*(\bar{k}-2)}]$$

$$\left[4\left(\sum_{\bar{k}+2<i} N_{g^*(i)} - \sum_{i<\bar{k}-2} N_{g^*(i)}\right) + \sum_{\bar{k}-2<i<\bar{k}+2} (2i - 2\bar{k})\, N_{g^*(i)}\right];$$

by Figure 5

$$= [N_4 - N_{g^*(\bar{k}-2)}]\left[4\left(\sum_{\bar{k}+2<i} N_{g^*(i)} - \sum_{i<\bar{k}-2} N_{g^*(i)}\right) + 2(N_3 - N_2)\right];$$

by (3.12) and (1.1)

$$< 0.$$

This contradicts the assumption that g^* is an optimal solution of problem (3.6). Hence $C_{g^*(\bar{k}-2)} = C_4$ as asserted.

The theorem can be proven by continuing using similar arguments.

4. Conclusion

Assuming that a decision-maker has decided upon using a linear communication system, this article shows how divided communication areas may be linked on a trunk in linear order to minimize the loss caused by communication delays due to communication flow increases. Generally speaking, whether the arrangement is good or not depends upon the level of service quality, while an important index of the level of service quality is the extent of communication delay. Since the total communication flow of the whole system is in direct proportion to the number of communication delays, we have established this mathematical model for studying the effects, on communication flow, of various arrangements of communication centres. Through careful deduction, the optimal arrangement for minimizing communication flow has been achieved, its features being : from the centre of the sequence line as a base, communication centres will be installed regularly and in order according to the communication demand of the users.

The above result of research could be used not only as reference criterion for linear communication system design, every branches of the hierarchical network structure could be also used topologically as well. For non-linear communications network topology, how can all divided communication areas be linked to a given network topology to minimize communication delay loss ? In addition, what is the optimal arrangement for a linear or non-linear network topology considering cost factors ? These problems are issues worth studying in the future.

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