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FIXED POINT THEOREMS FOR MULTIVALUED PROBABILISTIC (Ψ)-CONTRACTIONS

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Jin-Xuan Fang[2] proved a fixed point theorem for multivalued probabilistic (Ψ)-contraction on a Menger space $(S, \mathcal{F}, t)$, where t is of h-type. Here we prove two fixed point theorems for multivalued probabilistic (Ψ)-contraction $f : S \to \mathcal{P}(S)\backslash\theta$. In the first theorem t is a continuous T-norm (not necessarily of h-type) and in the second one Ψ satisfies some different conditions than in Fang's paper and t is of h-type.

1. INTRODUCTION

Some fixed point theorems for multivalued mappings in probabilistic metric spaces are proved in Fang[2]. T-norm t is such that $t(x, x) \geq x$, $x \in [0, 1]$, which implies that $t(a, b) = \min \{a, b\}$ $(a, b \in [0, 1])$ (see Mishra *et al.*[8], Singh and Pant[10]). In Fang[2] and Hadžić[3] T-norm t is of h-type and in Hadžić[5] t is a continuous T-norm.

Hadžić[5] used in proof β function of noncompactness introduced by Tan[11], which is a natural generalization of the notion of the measure of noncompactness.

Theorem 1, which we shall prove here, is a generalization of Theorem 1 of Hadžić[5]. In Theorem 2 T-norm t is of h-type. It will be shown that the (ε, λ)-topology in such an $(S, \mathcal{F}, t)$ can be introduced by the family $\{d_n\}_{n \in \mathbf{N}}$ of pseudometrics. Hence, we can prove Theorem 2 using a similar method as in the case of metric spaces.

2. PRELIMINARIES

By Δ we shall denote the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping of $\mathbf{R}$ into $[0, 1]$ so that $\sup_{x \in \mathbf{R}} F(x) = 1$). If $(S, \mathcal{F})$ is a probabilistic metric space[9] we shall denote $\mathcal{F}(p, q)$ $(p, q \in S)$ by

$F_{p,q}$. A Menger space is a triple $(S, \mathcal{F}, t)$ such that $(S, \mathcal{F})$ is a probabilistic metric space and t is a T-norm i.e. a mapping from $[0, 1] \times [0, 1]$ into $[0, 1]$ such that :

(i) $t(a, 1) = a$, for every $a \in [0, 1]$;

(ii) $t(a, b) = t(b, a)$, for every $a, b \in [0, 1]$;

(iii) For every $a, b, c, d \in [0, 1]$:

$a \geq b,\ c \geq d \Rightarrow t(a, c) \geq t(b, d)$;

(iv) $t(a, t(b, c)) = t(t(a, b), c)$, for every $a, b, c \in [0, 1]$.

and the following inequality holds

$$F_{p,q}(x+y) \geq t(F_{p,r}(x) F_{r,q}(y))$$

for all $p, q, r \in S$ and $x, y \in R$.

If T-norm t is continuous then the (ε, λ)-topology in $(S, \mathcal{F}, t)$, defined by the family $\{U_v(\varepsilon, \lambda) : v \in S,\ \varepsilon \in (0, \infty),\ \lambda \in (0, 1)\}$, is metrizable.

Here

$$U_v(\varepsilon, \lambda) = \{u;\ u \in S,\ F_{u,v}(\varepsilon) > 1 - \lambda\}.$$

If $(S, \mathcal{F})$ is a probabilistic metric space and $\Psi : [0, \infty) \to [0, \infty)$ we say that a mapping $f : S \to \mathcal{P}(S)\backslash\theta$ is 'a multivalued probabilistic' (Ψ)-contraction if for every $x, y \in S$ and every $P \in fx$ there exists $q \in fy$ such that

$$F_{p,q}(\Psi(\varepsilon)) \geq F_{x,y}(\varepsilon), \text{ for every } \varepsilon > 0.$$

Let G be the set of all nonnegative upper semicontinuous normal convex fuzzy numbers. If x is a fuzzy number, $[x]_\alpha = \{u;\ x(u) \geq \alpha\}$, $\alpha \in (0, 1]$. Let X be a nonempty set, $d : X \times X \to G, L$ and R symmetric mappings from $[0, 1] \times [0, 1]$ into $[0, 1]$ nondecreasing in both arguments such that $L(0, 0) = 0$, $R(1, 1) = 1$ and let $[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)]$; $x, y \in X$, $\alpha \in (0, 1]$. By $\bar{0}$ we denote the fuzzy number defined by $\bar{0}(u) = 1$ for $u = 0$ and $\bar{0}(u) = 0$ for $u \neq 0$.

The quadruple (X, d, L, R) is called 'a fuzzy metric space' and d 'a fuzzy metric' if and only if (a)-(c) hold, where :

(a) $d(x, y) = \bar{0} \Leftrightarrow x = y$.

(b) $d(x, y) = d(y, x)$, for all $x, y \in X$.

(c) $d(x, y)(s + u) \geq L(d(x, z)(s), d(z, y)(u))$ for all $x, y, z \in X$, whenever $s \leq \lambda_1(x, z)$, $u \leq \lambda_1(z, y)$, $s + u \leq \lambda_1(x, y)$;

$$d(x, y)(s + u) \leq R(d(x, z)(s), d(z, y)(u))$$

for all $x, y, z \in X$, whenever

$$s \geq \lambda_1(x, z),\ u \geq \lambda_1(z, y),\ s + u \geq \lambda_1(x, y).$$

Kaleva and Seikkala[7] proved some connections between fuzzy metric and probabilistic metric spaces.

If (X, d, L, R) is a fuzzy metric space such that R is associative, $R(a, 0) = a$, for every $a \in [0, 1]$, $R(a, 1) = R(1, a) = 1$, for every $a \in [0, 1]$ and $\lim_{u \to \infty} d(x, y)(u) = 0$, for all $x, y \in X$ then $(X, \mathcal{F}, t)$ is a Menger space, where (Kaleva and Seikkala[7]) :

$$t(a, b) = 1 - R(1 - a, 1 - b),\ a, b \in [0, 1],$$

$$F_{x,y}(s) = \begin{cases} 0, & s \leq \lambda_1(x, y) \\ 1 - d(x, y)(s), & s \geq \lambda_1(x, y). \end{cases}$$

If $(S, \mathcal{F})$ is a probabilistic metric space Tan[11] introduced the notion of a β_A function of noncompactness, for every probabilistic bounded subset $A \subset S$.

Definition 1 — Let A be a nonempty subset of S. The function $D_A(\cdot) : \mathbf{R}^+ \to [0, 1]$ defined by

$$D_A(u) = \sup_{s<u} \inf_{p,q \in A} F_{p,q}(s),\ u \in \mathbf{R}^+$$

is called the probabilistic diameter of the set A and A is probabilistic bounded if and only if $\sup_{u \in \mathbf{R}^+} D_A(u) = 1$.

Definition 2 — If A is a probabilistic bounded subset of S, the $\beta_A : \mathbf{R}^+ \to [0, 1]$ function of noncompactness is :

$\beta_A(u) = \sup \{\varepsilon \geq 0$, there exists a finite subset A_f of S such that $\tilde{F}_{A,A_f}(u) \geq \varepsilon\}$.

For every probabilistic bounded subsets $A, B \subset S$:

$$\tilde{F}_{A,B(u)} = \sup_{s<u} \inf_{x \in A} \sup_{y \in B} F_{x,y}(s).$$

For every probabilistic bounded subsets $A, B \subset S$ we have that (1)-(6) holds, where (Tan[11]) :

(1) $\beta_A \in \Delta$.

(2) $\beta_A(u) \geq D_A(u)$, for every $u \in \mathbf{R}$.

(3) $\phi \neq A \subset B \subset S \Rightarrow \beta_A(u) \geq \beta_B(u)$, for every $u \in \mathbf{R}$.

(4) $\beta_{A \cup B}(u) = \min \{\beta_A(u), \beta_B(u)\}$, for every $u \in \mathbf{R}$.

(5) $\beta_A(u) = \beta_{\bar{A}}(u)\ (u \in \mathbf{R})$, where $\bar{A}$ is the closure of A.

(6) $\beta_A = H$ (H is the function : $H(u) = 1$, for $u > 0$ and $H(u) = 0$, $u \leq 0$) $\Rightarrow A$ is precompact.

Let $(S, \mathcal{F})$ be a probabilistic metric space, K a probabilistic bounded subset of S and $f : K \to \mathcal{P}(S) \backslash \varphi$. If $f(K)$ is a probabilistic bounded subset of S and for every $B \subset K$:

$$\beta_{f(B)}(u) \leq \beta_B(u),\ \text{for every } u > 0 \Rightarrow B \text{ is precompact}$$

then f is 'densifying on the set K with respect to the function' β.

3. Fixed Point Theorems

Let $\mathcal{M} = \{\Psi;\ \Psi : [0, \infty) \overset{\text{onto}}{\to} [0, \infty),\ \Psi$ is strictly increasing and $\lim_{n\to\infty} \Psi^n(x) = 0$, for every $x \in [0, \infty)\}$. If $\Psi \in \mathcal{M}$ let $\Psi^{-1} = \bar{\Psi}$.

By $Com(M)$ and 2_c^M we shall denote the family of all nonempty, compact and closed subsets of M, respectively.

Theorem 1 — Let $(S, \mathcal{F}, t)$ be a complete Menger space with a continuous T-norm t, M a nonempty, closed and probabilistic bounded subset of S and $f : M \to Com(M)$ a probabilistic Ψ-contraction. If $\Psi \in \mathcal{M}$ then there exists $x \in M$ such that $x \in Tx$.

Proof : Let $x_0 \in M$ and $x_1 \in fx_0$. Since f is a probabilistic Ψ-contraction there exists $x_2 \in fx_1$ such that for every $s > 0$:

$$f_{x_2, x_1}(s) \geq F_{x_1, x_0}(\bar{\Psi}(s)).$$

Hence, we can define a sequence $\{x_n\}_{n\in\mathbf{N}}$ from M such that $x_{n+1} \in fx_n$ $(n \in \mathbf{N})$ and

$$f_{x_{n+1}, x_n}(s) \geq F_{x_1, x_0}(\bar{\Psi}^n(s)) \quad (n \in \mathbf{N}).$$

We shall prove that the set $\{x_n ; n \in \mathbf{N}\}$ is compact. This will imply that there exists a convergent subsequence $\{x_{n_k}\}_{k\in\mathbf{N}}$ of the sequence $\{x_n\}_{k\in\mathbf{N}}$.

By $\beta^* = \beta^M$ we shall denote the function β in the induced probabilistic metric space $(M, \mathcal{F}, t)$.

We shall prove that for every $s > 0$ and every $A \subset M$

$$\beta_{f(A)}(\Psi(s)) \geq \beta_A(s). \qquad \text{... (1)}$$

Since β is a left-continuous function it is enough to prove that for every $u \in (0, s)$

$$\beta_A(s-u) \leq \beta_{f(A)}(\Psi(s)). \qquad \text{... (2)}$$

If $\beta_A(s-u) = 0$ then (2) holds. So, we shall suppose that $\beta_A(s-u) > 0$. In order to prove (2) we shall prove the following implication.

$$r > 0,\ r < \beta_A(s-u) \in r \leq \beta_{f(A)}(\Psi(s)).$$

Let $0 < r < \beta_A(s-u)$. From the definition of $\beta_A(\cdot)$ we conclude that there exists a finite subset $A_f \subset M$ such that

$$\tilde{F}_{A, A_f}(s-u) > r$$

and so

$$\inf_{z \in A} \max_{w \in A_f} F_{z,w}(s-u) > r. \qquad \text{... (3)}$$

Hence, for every $z \in A$ there exists $w(z) \in A_f$ such that $f_{z,w(z)}(s-u) > r$. Let $y \in fz$, $z \in A$. Then there exists $x \in fw(z)$ such that

$$f_{y,x}(\Psi(s-u)) \geq F_{z,w(z)}(s-u) > r.$$

From the continuity of t and relation $t(r, 1) = r$, it follows that for every $\delta \in (0, r)$ there exists $\lambda(\delta) \in (0, 1)$ such that

$$1 \geq h > 1 - \lambda(\delta) \Rightarrow t(r, h) > r - \delta.$$

Let $A_f = \{x_1, x_2, ..., x_n\}$. From the compactness of fx_i $(i \in \{1, 2, ..., n\})$ it follows the existence of a finite set $\{z_1^i, z_2^i, ..., z_{n(i)}^i\} \subset M$ $(i \in \{1, 2, ..., n\})$ such that

$$f_{x_i} \subset \bigcup_{j=1}^{n(i)} U_{z_j^i}\left(\frac{\Psi(s)-\Psi(s-u)}{2}, \lambda(\delta)\right)$$

where $\quad U_v(\varepsilon, \lambda) = \{z, F_{z,v}(\varepsilon) > 1 - \lambda\}.$

We shall prove that

$$\tilde{F}_{f(A), \bigcup_{i=1}^{n} \bigcup_{j=1}^{n(i)} \{z_j^i\}}(\Psi(s)) > r - \delta$$

which means that

$$\sup_{\alpha < \Psi(s)} \inf_{y \in f(A)} \max_{w \in B_f} F_{y,w}(a) > r - \delta, \qquad \text{... (4)}$$

where $\quad B_f = \bigcup_{i=1}^{n} \bigcup_{j=1}^{n(i)} \{z_j^i\}.$

Let $y \in f(A)$ and $x \in f(A_f)$ such that

$$F_{y,x}(\Psi(s-u)) > r.$$

If $x \in f_{x_i}$ for some $i \in \{1, 2, ..., n\}$, then there exists z_j^i (for some $j \in \{1, 2, ..., n(i)\}$) such that $x \in U_{z_j^i}\left(\frac{\Psi(s)-\Psi(s-u)}{2}, \lambda(\delta)\right)$ and so

$$F_{x,z_j^i}\left(\frac{\Psi(s)-\Psi(s-u)}{2}\right) > 1 - \lambda(\delta). \qquad \text{... (5)}$$

From (5) we have that

$$F_{y,z_j^i}\left(\frac{\Psi(s)+\Psi(s-u)}{2}\right)$$

$$\geq t\left(F_{y,x}(\Psi(s-u)),\ F_{x,z_j^i}\left(\frac{\Psi(s)-\Psi(s-u)}{2}\right)\right) > r-\delta. \quad \ldots (6)$$

Relation (6) implies that

$$\sup_{\alpha<\Psi(s)}\ \inf_{y\in f(A)}\ \max_{w\in B_f}\ F_{y,w}(a)$$

$$\geq \inf_{y\in f(A)}\ \max_{w\in B_f} F_{y,w}\left(\frac{\Psi(s)+\Psi(s-u)}{2}\right) > r-\delta$$

and so (4) holds.

From (4) we obtain that

$$r-\delta \leq \beta_{f(A)}(\Psi(s))$$

for every $\delta\in(0,r)$ and since δ is an arbitrary element from $(0, r)$ we have that $r\leq\beta_{f(A)}(\Psi(s))$. Hence for every $s>0$ and every $A\subset M$:

$$\beta_{f(A)}(\Psi(s))\geq\beta_A(s). \quad \ldots (7)$$

Relation (7) implies that f is densifying on M in respect to β. Indeed, suppose that $\beta_{f(A)}(s)\leq\beta_A(s)$, for every $s>0$. Then from (7) we obtain that

$$\beta_{f(A)}(s)\geq\beta_{fA}(s)\geq\beta_A(\bar{\Psi}\ (s))\geq\ldots\geq\beta_A(\bar{\Psi}^n(s)). \quad \ldots (8)$$

Since $\lim\limits_{n\to\infty}\ \bar{\Psi}^n(s)=+\infty$ from (8) we obtain that $\beta_A(s)=1$, for every $s>0$. Hence $\beta_A=H$ which implies that $\bar{A}$ is compact.

Let $A=\{x_n;\ n\in\mathbf{N}\}$. Since $x_{n+1}\in fx_n\ (n\in\mathbf{N})$ we have that

$$\beta_{f[(x_n;\,n\geq1)]}(u)\ \leq\beta_{(x_n;\,n\geq2)}(u)\ =\ \beta_{(x_n;\,n\geq1)}(u)$$

and so

$$\beta_{f(A)}(u)\leq\beta_A(u).$$

Since f is densifying on M in respect to β it follows that $\{x_n;\ n\in\mathbf{N}\}$ is compact. Let $\lim\limits_{k\to\infty}\ x_{n_k}=x\in M$. From the inequality

$$F_{x_{n_k+1},\,x_{n_k}}(s)\ \geq F_{x_1,x_0}(\bar{\Psi}^{n_k}(s))\ (k\in\mathbf{N})$$

and the relation $\lim\limits_{k\to\infty}\ \bar{\Psi}^{n_k}(s)=\infty$ it follows that $\lim\limits_{k\to\infty}\ x_{n_k\oplus1}=x$ and since $x_{n_k+1}\in fx_{n_k}\ (k\in\mathbf{N})$ it remains to be proved that the mapping f is closed.

Let $\{v_n\}_{n\in\mathbf{N}}$ and $\{w_n\}_{n\in\mathbf{N}}$ be two sequences from M such that $w_n \in fv_n\,(n\in\mathbf{N})$ and $\lim_{n\to\infty} w_n = w$, $\lim_{n\to\infty} v_n = v$. We shall prove that $w\in \overline{f(v)}$ and since $\overline{f(v)} = f(v)$, this means that $w\in fv$. Let $\varepsilon > 0$ and $\delta \in (0, 1)$.

Since f is a probabilistic Ψ-contraction for every $n\in\mathbf{N}$ there exists $z_n \in fv$ such that

$$F_{z_n, w_n}(s) \geq F_{v_n, v}(\bar{\Psi}\ (s)), \text{ for every } s > 0.$$

Then

$$F_{w,z_n}(\varepsilon) \geq t\left(F_{w,w_n}\left(\frac{\varepsilon}{2}\right), F_{w_n,z_n}\left(\frac{\varepsilon}{2}\right)\right) \geq t\left(F_{w,w_n}\left(\frac{\varepsilon}{2}\right), F_{v_n,v}\left(\bar{\Psi}\left(\frac{\varepsilon}{2}\right)\right)\right)$$

From the continuity of the mapping t and the relation $t(1, 1) = 1$ it follows that there exists $u \in (0, 1)$ such that $t(s, s) > 1 - \delta$ for every $s \in (u, 1)$.

If for $n = n_0(\varepsilon, s)\in\mathbf{N}$ and $s \in (u, 1)$:

$$F_{w,w_n}\left(\frac{\varepsilon}{2}\right) > s \text{ and } F_{v_n,v}\left(\bar{\Psi}\left(\frac{\varepsilon}{2}\right)\right) > s$$

then $$F_{w,z_n}(\varepsilon) \geq t(s, s) > 1 - \delta$$

which means that $z_n \in U_w(\varepsilon, \delta)$. Hence $w\in \overline{fv}$ which completes the proof.

Similarly as in Hadžić[5] from Theorem 1 we obtain the following Corollary for fuzzy metric spaces.

The proof is similar to the proof of Theorem 2 of Hadžić[5].

We suppose that R is associative and $R(a, 0) = a$, for every $a\in [0, 1]$.

Corollary 1 — Let (X, d, L, R) be a fuzzy metric space such that $\lim_{u\to\infty} d(x,y)(u) = 0$, for all $x, y \in X$, $R(a, 1) = R(1, a) = 1$, for $a \in [0, 1]$, R is continuous and for every $\alpha\in (0, 1)$ there exists M_α so that $\sup_{x,y\in X} \rho_\alpha(x,y) < M_\alpha$. Let $f : X \to Com(X)$ and $\Psi\in\mathcal{M}$ be such that the following condition is satisfied :

For every $x, y \in X$ and every $p\in fx$ there exists $q\in fy$ such that

$$s \geq \lambda_1(x,y) \Rightarrow d(x,y)(s) \geq 1 - F_{p,q}(\Psi - (s)).$$

If X is complete as a Menger space there exists $x \in X$ such that $x\in fx$.

In the next Lemma T-norm t is of h-type which means that the family $\{T_n(x)\}_{n\in\mathbf{N}}$ is equicontinuous at the point $x = 1$, where

$$T_1(x) = t(x,x),\quad T_n(x) = t(x, T_{n-1}(x)),\quad n\geq 2$$

$(x\in [0, 1])$. If $t = \min$, t is of h-type. A nontrivial example of a t norm which is of h-type is given in Hadžić[3]. The following result is well known[6] : If t is a continuous T-norm and $I = [0, 1]$ then :

$$I \times I = \left(\bigcup_{k \in K} J_k \times J_k \right) \bigcup J,$$

where K is at most denumerable, for every $k \in K$, J_k is an open interval, $J_k \bigcap J_r = \phi$, for $k \neq r$, $t \mid J_k \times J_k = t_k$ is an Archimedean semigroup ($k \in K$) and $t \mid J = \min$.

In this case we shall use the notation

$$t = \{\langle J_k, t_k \rangle,\ k \in K\}.$$

Lemma 1 — Let $t = \{\langle J_k, t_k \rangle,\ k \in K\}$ be a continuous T-norm of h-type and for every $k \in K$, t_k is strict. If $(S, \mathcal{F}, t)$ is a Menger space then there exists a sequence $\{a_n\}_{n \in \mathbf{N}}$ from (0, 1) such that the family of pseudometrics $\{d_n\}_{n \in \mathbf{N}}$ defines the (ε, λ)-topology in S, where

$$d_n(x, y) = \sup\ \{t; F_{x,y}(t) \leq a_n\}\ \ (n \in \mathbf{N};\ x, y \in S). \qquad \dots (9)$$

PROOF : The proof of the Lemma in fact contained in a part of the proof of the Theorem of Hadžić[4], but we shall give it here for the sake of completeness. Suppose that for every $k \in K$, $J_k = (c_k, b_k)$. In the case when K is empty the proof is trivial since then $t = \min$. In this case $\{a_n\}_{n \in \mathbf{N}}$ is an arbitrary sequence from (0, 1) such that $\lim_{n \to \infty} a_n = 1$. It follows easily that for every $n \in \mathbf{N}$, d_n is a pseudometric. It is obvious that the family $\{d_n\}_{n \in \mathbf{N}}$ defines the (ε, λ)-topology.

Suppose that $\phi \neq K = \{1, 2, \dots, m\}$. If $b_m < 1$ then the restriction $t \mid [b_m, 1] \times [b_m, 1] = \min$ and we can take for $\{a_n\}_{n \in \mathbf{N}}$ an arbitrary sequence from the interval $(b_m, 1)$ such that $\lim_{n \to \infty} a_n = 1$. The relation $b_m = 1$ contradicts the condition that t is of h-type. Namely, Cho-Hsin Ling[6] proved that $t_m = t \mid [c_m, b_m] \times [c_m, b_m]$ is such that for every $x \in (c_m, 1)$: $\lim_{n \to \infty} T_n(x) = c_m$. Hence, if $\lambda \in (0, 1)$ is such that $1 - \lambda > c_m$ then there is no $\eta(\lambda) \in (0, 1)$ such that the following implication holds :

$$x > \eta(\lambda) \Rightarrow T_n(x) > 1 - \lambda,$$

which is a contradiction.

Let K be infinite. If $b_m \leq b < 1$, for every $m \in K$ then we can take again for $\{a_n\}_{n \in \mathbf{N}}$ an arbitrary sequence, such that $a_n > b$, $n \in \mathbf{N}$ and $\lim_{n \to \infty} a_n = 1$. In the case that there is no $b \in (0, 1)$ such that $b_m \leq b$, for every $m \in \mathbf{N}$ then there exists a subsequence $\{c_{m(n)}\}_{n \in \mathbf{N}}$ such that $\lim_{n \to \infty} c_{m(n)} = 1$. Then by definition $a_n = c_{m(n)}$, for every $n \in \mathbf{N}$.

It is easy to prove that the family of pseudometric (9) defines the (ε, λ)-topology, since $t(a_n, a_n) = a_n$, for every $n \in \mathbf{N}$ and for every $k \in K$ the semigroup t_k is strict.

Example — If $t \approx \{\langle (1-2^{-k}, 1-2^{-k-1}), \bar{t} \rangle, k \in \mathbf{N}\}$ then t is of h-type (here $\bar{t}$ is an arbitrary T-norm).

Theorem 2 — Let $(S, \mathcal{F}, t)$ be a complete Menger space, M a nonempty and closed subset of S and $f : M \to 2_c^M$ a probabilistic (Ψ)-contraction such that the following conditions are satisfied :

(a) t is a continuous T-norm of h-type, $t \approx \{\langle J_k, t_k \rangle k \in K\}$, t_k is strict $(k \in K)$.

(b) $\Psi : [0, \infty) \to [0, \infty)$ is continuous from the right, $\Psi(t) < t$, for every $t > 0$ and

$$\lim_{t \to 0^+} \sup \left[\frac{\Psi(t)}{t} \right] = q < 1.$$

Then there exists a fixed point of the mapping f.

PROOF : By the Lemma there exists a sequence $\{a_n\}_{n \in \mathbf{N}}$ from $(0, 1)$ such that $\lim_{n \to \infty} a_n = 1$ and that the family $\{d_n\}_{n \in \mathbf{N}}$ given by (9) defines the (ε, λ)-topology on S.

We shall prove that the condition that f is a probabilistic (Ψ)-contraction implies the following condition :

For every $x, y \in M$ and every $p \in fx$ there is a $q \in fy$ such that for every $n \in \mathbf{N}$

$$d_n(p, q) \leq \Psi(d_n(x, y)) \qquad \text{... (10)}$$

Let $x, y \in M$ and $p \in fx$. Since f is a probabilistic (Ψ)-contraction there exists $q \in fy$ such that for every $\delta > 0$:

$$F_{p,q}(\Psi(\delta)) \geq F_{x,y}(\delta). \qquad \text{... (11)}$$

We shall prove that for every $\delta > 0$ the following implication holds for every $n \in \mathbf{N}$

$$d_n(x, y) < \delta \Rightarrow d_n(p, q) < \Psi(\delta). \qquad \text{... (12)}$$

If $d_n(x, y) < \delta$ from the definition of d_n we have that $F_{x,y}(\delta) > a_n$ and (11) implies that $F_{p,q}(\Psi(\delta)) > a_n$ i.e. $d_n(p, q) < \Psi(\delta)$. Since Ψ is continuous from the right and δ is an arbitrary real number such that $d_n(x, y) < \delta$ we obtain that (10) holds.

Using (10) we can construct a sequence $\{x_m\}_{m \in \mathbf{N}}$ from M such that $x_{m+1} \in fx_m$ $(m \in \mathbf{N} \cup \{0\})$ and

$$d_n(x_{m+1}, x_m) \leq \Psi(d_n(x_m, x_{m-1})), \quad m \in \mathbf{N}, \; n \in \mathbf{N}.$$

We shall prove in a usual way[1] that $\{x_m\}_{m\in\mathbf{N}}$ is a Cauchy sequence which means that for every $\varepsilon > 0$ and $n \in \mathbf{N}$ there exists $r_0(\varepsilon, n) \in \mathbf{N}$ such that

$$d_n(x_r, x_{r+p}) < \varepsilon,\ r \geq r_0(\varepsilon, n), p \in \mathbf{N}. \qquad \text{... (13)}$$

Let for every $m, n \in \mathbf{N}$:

$$c_m^n = d_n(x_m, x_{m-1}).$$

If for some $n \in \mathbf{N}$ and $m \in \mathbf{N}$, $c_m^n = 0$ from

$$c_{m+1}^n \leq \Psi(c_m^n)$$

using $\Psi(0) = 0$, we obtain that $c_r^n = 0$ for every $r > m$ and then

$$d_n(x_r, x_{r+p}) \leq \sum_{s=r+1}^{r+p} c_s^n = 0 < \varepsilon,\ r > m,\ p \in \mathbf{N}.$$

Hence, for such an $n \in \mathbf{N}$ (13) is satisfied. Suppose that $c_m^n > 0$ for every $m \in \mathbf{N}$. Then $\Psi(c_m^n) < c_m^n$ and from $c_{m+1}^n < c_m^n$ it follows the existence of $c^n = \lim_{m\to\infty} c_m^n$. Then $c^n = 0$ since $c^n > 0$ implies that $c^n \leq \Psi(c^n) < c^n$, which is a contradiction. Further, we have that :

$$d_n(x_r, x_{r+p}) \leq c_{r+1}^n + c_{r+2}^n + ... + c_{r+p}^n \leq \sum_{k=r+1}^{\infty} c_k^n.$$

Since $c_m^n > 0$, for all $m \in \mathbf{N}$ and $\lim_{m\to\infty} c_m^n = c^n = 0$ we have that

$$\lim_m \sup \frac{c_{m+1}^n}{c_m^n} \leq \lim_{m\to\infty} \sup \frac{\Psi(c_m^n)}{c_m^n} = q < 1.$$

Hence $\sum_{m=1}^{\infty} c_m^n$ converges and so (13) holds.

If $\lim_{m\to\infty} x_m = x$ then $x \in fx$, which can be proved as in Theorem 1.

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