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ON EQUAL SUMS OF SIXTH POWERS

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A method of generating parametric solutions of the Diophantine equation $x^6+y^6+z^6=u^6+v^6+w^6$ is described. The method is used to obtain a new parametric solution of this equation.

§1. In this paper we shall be concerned with the Diophantine equation

$$x^6+y^6+z^6=u^6+v^6+w^6. \qquad \dots (1)$$

Subba Rao[5] gave a numerical solution of (1). Brudno[2] first gave a method of generating numerical solutions of (1) and later, he[3] obtained a two-parameter solution in terms of homogeneous polynomials of degree four. Bremner[1] has studied the equation in detail and has given two solutions of degree five and one solution of degree six.

We shall give an elementary method of obtaining a family of parametric solutions of (1) such that any parametric solution of the family generates another such solution. In particular, we shall obtain a new solution of degree eight in two parameters.

§2. Bremner[1] (p. 545) has shown that any solution of the system of equations

$$x^2+xu-u^2=w^2+wz-z^2 \qquad \dots (2)$$

$$y^2+yv-v^2=u^2+ux-x^2 \qquad \dots (3)$$

$$z^2+zw-w^2=v^2+vy-y^2 \qquad \dots (4)$$

also gives solutions of

$$x^r+y^r+z^r=u^r+v^r+w^r, \quad r = 2,\ 6. \qquad \dots (5)$$

We shall therefore obtain solutions of the system of eqns. (2), (3) and (4). Let $x=\alpha_1$, $y=\beta_1$, $z=\gamma_1$, $u=\alpha_2$, $v=\beta_2$, $w=\gamma_2$ be a given solution of this system. To obtain another solution, we write

$$\left.\begin{aligned} &x=a\theta+\alpha_1,\ u=d\theta+\alpha_2\\ &y=b\theta+\beta_1,\ v=e\theta+\beta_2\\ &z=c\theta+\gamma_1,\ w=\gamma_2. \end{aligned}\right\} \qquad \dots (6)$$

We may write (2) as

$$(x-w)(2x+z+u+2w)=(z-u)(x-2z-2u+w)$$

or,

$$\frac{x-w}{z-u}=\frac{x-2z-2u+w}{2x+z+u+2w}=s$$

where s is arbitrary. Thus, eqn. (2) is equivalent to the two linear equations :

$$(x-w)-s(z-u)=0$$

$$(x-2z-2u+w)-s(2x+z+u+2w)=0.$$

Now, on making the substitutions (6), taking $s=(\alpha_1-\gamma_2)/(\gamma_1-\alpha_2)$ and using the known relation $\alpha_1^2+\alpha_1\alpha_2-\alpha_2^2=\gamma_2^2+\gamma_1\gamma_2-\gamma_1^2$, we find that for $\theta\neq 0$, (2) reduces to the following two equations :

$$(\gamma_1-\alpha_2)a-(\alpha_1-\gamma_2)(c-d)=0 \qquad \ldots (7)$$

$$(\gamma_1-\alpha_2)(a-2c-2d)-(\alpha_1-\gamma_2)(2a+c+d)=0. \qquad \ldots (8)$$

Similarly, (4) may be written as

$$\frac{z-v}{y-w}=\frac{-2y+z+v-2w}{y+2z+2v+w}=t$$

where t is arbitrary. On making the substitutions (6), taking $t=(\gamma_1-\beta_2)/(\beta_1-\gamma_2)$ and using the known relation $\gamma_1^2+\gamma_1\gamma_2-\gamma_2^2=\beta_2^2+\beta_1\beta_2-\beta_1^2$, we find, as before, that (4) reduces to the following two linear equations :

$$(\beta_1-\gamma_2)(c-e)-(\gamma_1-\beta_2)b=0 \qquad \ldots (9)$$

$$(\beta_1-\gamma_2)(-2b+c+e)-(\gamma_1-\beta_2)(b+2c+2e)=0. \qquad \ldots (10)$$

Equations (7)-(10) are linear in a, b, c, d, e and may accordingly be solved. The solution is given by

$$a=\{(\alpha_1-\gamma_2)^2+2(\alpha_1-\gamma_2)(\gamma_1-\alpha_2)\}\,\{(\beta_1-\gamma_2)^2+(\beta_1-\gamma_2)(\gamma_1-\beta_2)-(\gamma_1-\beta_2)^2\},$$

$$b=\{(\beta_1-\gamma_2)^2-2(\beta_1-\gamma_2)(\gamma_1-\beta_2)\}\,\{(\gamma_1-\alpha_2)^2+(\gamma_1-\alpha_2)(\alpha_1-\gamma_2)-(\alpha_1-\gamma_2)^2\},$$

$$c=\{(\gamma_1-\alpha_2)^2+(\gamma_1-\alpha_2)(\alpha_1-\gamma_2)-(\alpha_1-\gamma_2)^2\}$$
$$\times\{(\beta_1-\gamma_2)^2+(\beta_1-\gamma_2)(\gamma_1-\beta_2)-(\gamma_1-\beta_2)^2\},$$

$$d=-\{(\gamma_1-\alpha_2)^2+(\alpha_1-\gamma_2)^2\}\,\{(\beta_1-\gamma_2)^2+(\beta_1-\gamma_2)(\gamma_1-\beta_2)-(\gamma_1-\beta_2)^2\},$$

$$e=\{(\gamma_1-\beta_2)^2+(\beta_1-\gamma_2)^2\}\,\{(\gamma_1-\alpha_2)^2+(\gamma_1-\alpha_2)(\alpha_1-\gamma_2)-(\alpha_1-\gamma_2)^2\}. \qquad \ldots (11)$$

With these values of a, b, c, d, e and x, y, z, u, v, w given by (6), eqns. (2)

and (4) hold for all values of θ while (3) reduces to a linear equation in θ whose solution is given by

$$\theta = -\frac{a(2\alpha_1-\alpha_2)+b(2\beta_1+\beta_2)-d(\alpha_1+2\alpha_2)+e(\beta_1-2\beta_2)}{a^2-ad+b^2+be-d^2-e^2}$$

This leads to the following solution of the system of eqns. (2), (3) and (4) :

$$\left.\begin{aligned} x &= ap+\alpha_1 q,\ u = dp+\alpha_2 q \\ y &= bp+\beta_1 q,\ v = ep+\beta_2 q \\ z &= cp+\gamma_1 q,\ w = \gamma_2 q \end{aligned}\right] \quad \ldots (12)$$

where

$$p = -\{a(2\alpha_1-\alpha_2)+b(2\beta_1+\beta_2)-d(\alpha_1+2\alpha_2)+e(\beta_1-2\beta_2)\}$$

$$q = a^2-ad+b^2+be-d^2-e^2$$

and a, b, c, d, e are as in (11). The solution given by (12) also satisfies the system (5).

Thus, given a solution of the system of equations (2), (3) and (4), we can find another solution of this system. We may next use the new solution just obtained to generate yet another solution of this system of equations and this process can be repeated to obtain a family of solutions. If the initial solution is parametric, the solutions generated by this parametric solution will also be parametric.

For a numerical example, we must start with a known solution of eqns. (2), (3) and (4). Bremner[1] (p. 545) has pointed out that the table of solutions of (1) listed by Lander *et al.* (p. 456) gives (with one exception) after suitable relabelling and sign-changing, solutions of (2), (3) and (4). Thus, we may start with the following as a given solution of (2), (3) and (4) :

$$x = 3,\ y = 22,\ z = -19,\ u = 23,\ v = 15,\ w = 10.$$

Then we get the new solution

$$x = 233,\ y = 916,\ z = -711,\ u = 939,\ v = 509,\ w = 508.$$

Similarly, starting from the known solution

$$x = 74,\ y = 47,\ z = 33,\ u = 23,\ v = -54,\ w = 73$$

we get the new solution

$$x = -12788,\ y = -4531,\ z = -9407,\ u = 23,\ v = 11500,$$

$$w = -11845.$$

For obtaining parametric solutions, we start with the following parametric solution of eqns. (2), (3) and (4) :

$$\alpha_1 = -\gamma_2 = 2m-1$$

$$\beta_1 = +\beta_2 = -m^2 + m + 1$$

$$\gamma_1 = -\alpha_2 = m^2 + 1.$$

This solution of eqns. (2), (3) and (4) is easily obtained and is, in fact, a trivial solution of the system (1), but it leads to the following non-trivial solution of system of eqns. (2), (3) and (4) :

$$x = 2m^6 + 2m^5 + m^4 - 16m^3 + 35m^2 - 12m + 3$$

$$y = -m^6 + 3m^5 - 16m^4 + 27m^3 - 4m^2 - 9m + 5$$

$$z = 3m^6 - 2m^5 + 6m^4 + 16m^3 - 18m^2 + 14m + 1$$

$$u = -m^6 - 10m^5 + 14m^4 - 16m^3 - 2m^2 - 10m + 5$$

$$v = -3m^6 + m^5 - 4m^4 + 13m^3 + 4m^2 + m + 3$$

$$w = 2m^6 - 6m^5 - 11m^4 + 16m^3 - 25m^2 + 20m - 1.$$

This solution is, in effect, the same as the solution of degree 6 given by Bremner[1] (p. 575). We could use this solution to generate another parametric solution of eqns. (2), (3) and (4) but the computations being tedious are omitted.

As another example of a parametric solution, we may start with the following solution of degree 4 of eqns. (2), (3) and (4) given by Bremner[1] (p. 574) :

$$\alpha_1 = 3m^4 + 8m^3 + 9m^2$$

$$\beta_1 = -m^4 + 3m^3 + 14m^2 + 15m + 9$$

$$\gamma_1 = 2m^4 + 12m^3 + 19m^2 + 18m + 9$$

$$\alpha_2 = -2m^4 - 4m^3 + 5m^2 + 12m + 9$$

$$\beta_2 = 3m^4 + 9m^3 + 18m^2 + 21m + 9$$

$$\gamma_2 = m^4 + 10m^3 + 17m^2 + 12m.$$

This solution generates the following new solution of (1) :

$$x = -m^8 - 6m^7 - 20m^6 - 44m^5 - 58m^4 - 18m^3 + 49m^2 + 60m + 18$$

$$y = m^8 + 7m^7 + 19m^6 + 18m^5 - 2m^4 + 2m^3 + 37m^2 + 51m + 27$$

$$z = m^6 - 8m^5 - 52m^4 - 102m^3 - 84m^2 - 24m + 9$$

$$u = -4m^7 - 23m^6 - 50m^5 - 46m^4 + 8m^3 + 54m^2 + 54m + 27$$

$$v = -m^8 - 5m^7 - 7m^6 + 18m^5 + 74m^4 + 98m^3 + 59m^2 + 15m + 9$$

$$w = -m^8 - 8m^7 - 22m^6 - 30m^5 - 20m^4 - 8m^3 - 17m^2 - 36m - 18.$$

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