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BILINEAR AND TRILINEAR PARTITIONS OF A GRAPH

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Let $G = (V, E)$ be a graph. The bilinear partition number $\chi_{2L}(G)$ of G is the minimum order of a partition L of V such that for any two sets V_i and V_j in L, the subgraph $\langle V_i \bigcup V_j \rangle$ induced by $V_i \bigcup V_j$ is a linear forest. The trilinear partition number $\chi_{3L}(G)$ of G is the minimum order of a partition L of V such that for any three sets V_i, V_j and V_k in L, the subgraph $\langle V_i \bigcup V_j \bigcup V_k \rangle$ is a linear forest.

Theorem 1 — If $\Delta \geq 3$ is the maximum degree of G, then, $1 + \lceil \Delta/2 \rceil \leq \chi_{2L}$, with equality for trees. If G is a cactus, $\chi_{2L} \leq 2 + \lceil \Delta/2 \rceil$.

Theorem 2 — If every block of G is complete, and $\omega(G)$ is the clique number, then $\chi_{2L}(G) = \max \{\omega(G), 1 + \lceil \Delta/2 \rceil\}$.

Theorem 3 — If G is K_3-free, then $1 + \Delta \leq \chi_{3L}$, with equality for trees. Further, if G is a cactus, $\chi_{3L} \leq 2 + \Delta$.

We consider only finite, simple and undirected graphs, and follow the notations and terminology of the book by Harary[3].

Let $G = (V, E)$ be a graph. A 'bilinear partition' ($2L$-partition) of G is a partition $L = \{V_1, V_2, ..., V_n\}$ of V such that for all V_i and V_j in L, $i \neq j$, every component in the subgraph $\langle V_i \bigcup V_j \rangle$ induced by $V_i \bigcup V_j$ is either trivial or a path. Similarly, L is 'trilinear partition' ($3L$-partition) of G if for all distinct V_i, V_j and V_k in L, every component in the subgraph $\langle V_i \bigcup V_j \bigcup V_k \rangle$ is either trivial or a path.

The 'bilinear partition number' $\chi_{2L}(G)$ of G is the minimum order of a $2L$-partition of G. Such a partition is called a χ_{2L}-partition. Similarly, we define the 'trilinear partition number' $\chi_{3L}(G)$ and χ_{3L}-partition of G. We can regard the sets in L as colour classes and define $\chi_{2L}(G)$ as the minimum number of colours needed to colour the vertices of G such that the subgraph induced by the union of any two colour classes consists of only K_1 or paths as components. Similarly, $\chi_{3L}(G)$ can also be defined.

We observe that $\chi_{3L}(G)$ exists if, and only if, G has no triangle. One may be tempted to define similarly the numbers χ_{rL}, $r \geq 4$. These numbers do not exist for graphs with maximum degree at least 3. Hence, we concentrate on the numbers χ_{2L} and χ_{3L} in this paper.

The number χ_{2L} — We start with some elementary observations.

Proposition 1 — (i) $\chi_{2L}(K_n) = n$, (ii) $\chi_{2L}(K_{m,n}) = \lceil n/2 \rceil + m$, $1 \leq m \leq n$.

(iii) $\chi_{2L}(C_n) = 3$, $n \geq 3$, (iv) $\chi_{2L}(P_n) = 1 = \chi_{2L}(\overline{K}_n)$.

(v) For the wheel $W_n = C_n + K_1$, $n \geq 5$, $\chi_{2L}(W_n) = 1 + \lceil n/2 \rceil$.

(vi) For the complete multipartite graph $G = K_{P_1, P_2, \ldots, P_n}$, where

$$p_1 \leq p_2 \leq .. \leq p_n, \quad \chi_{2L}(G) = \lceil p_n/2 \rceil + \sum_{i=1}^{n-1} p_i .$$

We now obtain some bounds for χ_{2L}.

For any $v \in V$, let $N(v) = \{u \in V : uv \in E\}$, and $N[v] = N(v) \cup \{v\}$.

Proposition 2 — If $\Delta \geq 3$ is the maximum degree of G, then,

$$1 + \left\lceil \frac{\Delta}{2} \right\rceil \leq \chi_{2L}. \qquad \ldots (1)$$

PROOF : Let v be a vertex with deg $v = \Delta$, and $L = \{V_1, V_2, \ldots, V_n\}$ be a χ_{2L}-partition of V. Clearly, no four vertices including v in $N[v]$ can belong to the union $V_i \cup V_j$ of two sets in L, for otherwise, $K_{1,3}$ will be a subgraph in $\langle V_i \cup V_j \rangle$. Now, there are only two other possibilities for v and its neighbours.

1. The vertex v and exactly one of its neighbours belong to a set in L. In this case, all other neighbours of v should belong to different sets in L.

2. The set in L containing v does not contain any neighbour of v. In this case, at most two neighbours of v can belong to a single set in L.

In the first case, v and its neighbours belong to Δ-sets in L. In the second case, they belong to $1 + \lceil \Delta/2 \rceil$ sets.

Hence (1) follows.

As we see later, the bound in (1) is attained for almost all trees.

The clique number $\omega(G)$ of a graph G is the maximum number of vertices in a set S such that the subgraph $\langle S \rangle$ is complete.

Proposition 3 — Let G be a connected graph in which every block is complete.

Then $$\chi_{2L}(G) = \max \left\{ \omega(G), 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil \right\}. \qquad \ldots (2)$$

PROOF : Let $r = \max \left\{ \omega(G), 1 + \left\lceil \frac{\Delta(G)}{2} \right\rceil \right\}$, where G is as stated. Clearly, for any graph G, $\omega(G) \leq \chi_{2L}(G)$, since the vertices in any clique should belong to different colour classes in any $2L$-colouring of G. Now, from (1), $r \leq \chi_{2L}(G)$. We now prove $\chi_{2L} \leq r$ by describing a $2L$-colouring of G with r colours, and then (2) follows.

Clearly, the result holds if G has no cut vertex. Suppose now G has cut vertices, and $B_1, B_2, ..., B_k$ are the blocks of G at a cut vertex v. First colour the vertices of B_1 differently with r colours. If some more colours are left, colour the vertices in B_2 differently with the remaining colours. Continue this process of colouring the vertices in B_1, B_2, .. till all the r colours are exhausted, after colouring all or some vertices in B_i, $1 \leq i \leq k$. Now, colour the remaining vertices in B_i (if any) differently with a set S_1 of colours different from that already used in B_i. Next, colour the vertices in B_{i+1} differently with a set S_2 of colours different from S_1. Again colour the vertices differently in B_{i+2} with a set of colours different from $S_1 \bigcup S_2$. Continue this process till all the vertices in $B_1 \bigcup B_2 \bigcup .... \bigcup B_k$ are coloured.

This can be done, since first after colouring r vertices in $B_1 \bigcup B_2 \bigcup .... \bigcup B_i$, we are left with $n = \deg v - (r - 1)$ vertices to be coloured, and $n < r$. Next, let u be any other cut vertex in some block, say, B_1. Suppose, $B_1, U_1 U_2, ..., U_m$ are the blocks at u. Repeat the same process of colouring these blocks, noting that the block B_1 is already coloured, and continue this process till all the blocks of G are coloured.

In the above colouring, we observe that (i) no two adjacent vertices are coloured the same, and (ii) any vertex v with colour i is adjacent to at most two vertices with colour j, $i \neq j$. Thus, if V_i and V_j are two colour classes, the degree of every vertex in the subgraph $\langle V_i \bigcup V_j \rangle$ is at most two. Also, since the sets V_i and V_j are independent, and vertices in V_i as well as in V_j belong to different blocks of G, it follows that the subgraph $\langle V_i \bigcup V_j \rangle$ cannot contain a cycle, and hence it is a linear forest. This proves that the above colouring of G with r colours is a $2L$-colouring.

Corollary 3.1 — If G is a tree with $\Delta \geq 3$, then,

$$\chi_{2L}(G) = 1 + \left\lceil \frac{\Delta}{2} \right\rceil. \qquad ... (3)$$

A cactus is a connected graph C whose blocks are either K_2 or cycles. We now provide a tight upper bound for $\chi_{2L}(C)$.

Proposition 4 — For any cactus C there exists a $2L$-colouring with $s = 2u + \left\lceil \frac{\Delta(C)}{2} \right\rceil$ colours such that adjacent vertices are coloured differently.

PROOF (By induction on the number n of blocks in C) : When C is K_2 or a cycle, the result is true. Suppose it is true for all cactus with n blocks, and C is a cactus with $n + 1$ blocks. Let B be an end block at a cut vertex v. Suppose F is the graph obtained from C by removing all vertices of B except v. Since F has n blocks, by inductive hypothesis, there exists a $2L$-colouring of F with $2 + \left\lceil \frac{\Delta(F)}{2} \right\rceil$ and hence with s colours such that adjacent vertices are coloured differently. Consider such a colouring of F with s colours. Then the neighbours of v are coloured with at most $s - 1$ colours. Since we can give two nonadjacent neighbours of v the same colour, and

$$2(s-1) - \deg_F v \geq 2(s-1) - (\Delta(C) - 1) \geq 3,$$

it follows that we can colour at least three new neighbours of v with the same set of $s - 1$ colours. For this we need at least two distinct colours. Thus, after colouring all the neighbours of v in F, we are left with two distinct colours to colour new neighbours of v. Using these colours, we can colour the vertices in B (other than v) so that no two adjacent vertices are coloured the same, and thus obtain a $2L$-colouring of C with s colours.

From the result (1) and Proposition 4, we have

Corollary 4.1 — For any cactus, C,

$$1 + \left\lceil \frac{\Delta}{2} \right\rceil \leq \chi_{2L} \leq 2 + \left\lceil \frac{\Delta}{2} \right\rceil.$$

Note that the lower bound is attained for trees with $\Delta \geq 3$, and the upper bound is attained for cycles.

We now obtain some upper bounds for χ_{2L}.

One can easily prove :

Proposition 5 — Let G be a connected graph of order p. Then, $\chi_{2L}(G) \leq p$, with equality if, and only if, $G = K_p$.

Proposition 6 — Let α_0 and β_0 respectively denote the vertex covering and independence numbers of G. Then,

$$\chi_{2L}(G) \leq \alpha_0 + \left\lceil \frac{\beta_0}{2} \right\rceil. \qquad \dots (4)$$

PROOF : Let S be an independent set of vertices with $|S| = \beta_0$. Then $U = V - S$ is a minimum vertex cover. Partition S into $\left\lceil \frac{\beta_0}{2} \right\rceil$ sets, each containing at most two vertices, and partition U into α_0 singleton sets. Clearly, these $\alpha_0 + \left\lceil \frac{\beta_0}{2} \right\rceil$ sets form a $2L$-partition of $V(G)$, and (4) follows.

The bound in (4) can be improved in some cases.

Let α_1 and β_1 denote respectively the edge-covering and edge-independence numbers of a graph G of order p.

Proposition 7 — Let D be a minimum vertex cover of G, and $\beta_0(\langle D \rangle) = k$. (i) If G does not have the cycle C_4, then

$$\chi_{2L}(G) \leq (\alpha_0 - k) + \left\lceil \frac{k}{2} \right\rceil + \left\lceil \frac{\beta_0}{2} \right\rceil. \qquad \ldots (5)$$

(ii) if G does not have the cycles C_3 and C_4, then

$$\chi_{2L}(G) \leq \alpha_1. \qquad \ldots (6)$$

PROOF : (i) Let $r = \alpha_0 - k$, $n = \left\lceil \frac{k}{3} \right\rceil$ and $m = \left\lceil \frac{\beta_0}{2} \right\rceil$. The set $S = V - D$ is a maximum independent set of vertices, and let $\{V_1, V_2, ..., V_m\}$ be a partition of S into m sets, each set having at most two vertices. Let E be an independent set of k vertices in the subgraph $\langle D \rangle$, and $\{U_1, U_2, ..., U_n\}$ be a partition of E into n sets, each set having at most two vertices. Suppose $\{S_1, S_2, ..., S_r\}$ is a partition of D-E into singleton sets. Since the graph G is C_4-free, it is easy to verify that $\{S_1, ..., S_r, U_1, ..., U_n, V_1, ..., V_m\}$ is a $2L$-partition of $V(G)$, and (5) follows.

(ii) Let $B = \{e_1, e_2, ..., e_{\beta_1}\}$ be an independent set of edges, and $e_i = u_i v_i$, and $F_i = \{u_i, v_i\}$, $1 \leq i \leq \beta_1$. Further, let $\{A_1, A_2, ..., A_{p-2\beta_1}\}$ be a partition of $V(G) - (F_1 \bigcup F_2 \bigcup \cdots \bigcup F_{\beta_1})$ into singleton sets. Since G does not have the cycles C_3 and C_4, it is easy to verify that $\{A_1, A_2, ..., A_{p-2\beta_1}, F_1, F_2, ..., F_{\beta_1}\}$ is a $2L$-partition of $V(G)$. Hence $\chi_{2L}(G) \leq (p - 2\beta_1) + \beta_1 = p - \beta_1 = \alpha_1$.

The 'vertex linear arboricity' $\chi_L(G)$ of a graph G is the minimum order of a partition P of $V(G)$ such that for each V_i in P every component in the subgraph $\langle V_i \rangle$ is either trivial or a path (see Matsumoto[2]).

Proposition 8 — For any graph G,

$$\text{(i) } \chi_L(G) \leq \left\lceil \frac{\chi_{2L}(G)}{2} \right\rceil \qquad \ldots (7)$$

$$\text{(ii) } \chi(G) \leq 2 \left\lceil \frac{\chi_{2L}(G)}{2} \right\rceil. \qquad \ldots (8)$$

PROOF : (i) Let $L = \{V_1, V_2, ..., V_k\}$ be a χ_{2L}-partition of $V(G)$. Consider the partition $L_1 = \{V_1 \bigcup V_2, V_3 \bigcup V_4, ...\}$. Each set in L_1 induces a linear forest. Hence (7) follows.

(ii) Since the chromatic number of the subgraph induced by any set in L_1 is at most two, we have (8).

The number χ_{3L}. Let $G = (V, E)$ be a graph. The trilinear partition number $\chi_{3L}(G)$ of G is the minimum order of a $3L$-partition of V. As noted earlier, this number does not exist if G has a triangle. One can easily verify the following :

$$\chi_{3L}(K_{m,n}) = m + n, \text{ where } m + n \geq 3,$$

$$\chi_{3L}(C_n) = 4, \text{ and } \chi_{3L}(P_n) = \chi_{3L}(\bar{K}_n) = 1.$$

We now obtain some bounds for χ_{3L} which are analogous to those of χ_{2L}.

Proposition 9 — If $\Delta \geq 3$ is the maximum degree of a triangle-free graph G, then

$$1 + \Delta \leq \chi_{3L}(G). \quad \ldots (9)$$

PROOF : Let v be a vertex of G with deg $v = \Delta$. No two vertices in $N[v]$ can belong to the same set in any $3L$-partition L of $V(G)$, since otherwise the union of some three sets in L will induce a graph containing $K_{1,n}$, $n \geq 3$. This implies that there are at least $1 + \Delta$ sets in L, and (9) follows.

The bound in (9) is attained for all trees except paths.

Proposition 10 — For any tree T which is not a path,

$$\chi_{3L}(T) = 1 + \Delta. \quad \ldots (10)$$

PROOF : Let deg $v = \Delta$ in T. Colour all vertices in $N[V]$ with $\Delta + 1$ colours assigning different colours to all the vertices. Next consider any neighbour, say v_1 of v coloured 1. Suppose v is given the colour k. Since deg $v_1 \leq \Delta$, neighbours of v_1 can be coloured using at most Δ colours, where v_1 and all of its neighbours receive different colours. This process is continued till all vertices of T are coloured.

Consider the union of any three colour classes $V_i \bigcup V_j \bigcup V_k$. In the subgraph $H = \langle V_i \bigcup V_j \bigcup V_k \rangle$ each vertex of any one colour has at most one vertex of the other colours adjacent to it. Hence every vertex in the subgraph H has degree at most two, and since T is a tree, H is a linear forest. Therefore, we have $\chi_{3L}(T) \leq \Delta + 1$, and (10) follows by (9).

Proposition 11 — For a K_3-free cactus C, $\chi_{3L} \leq 2 + \Delta$.

PROOF : (By induction on the number n of blocks in C) : If $n = 1$, the result is true. Suppose it is true for all K_3-free cactus with n blocks, and let C be such a cactus with $n + 1$ blocks. Let v, B and F be as in the proof of Proposition 4. Suppose B is a cycle. Then, in F, deg $v \leq \Delta - 2$. Consider a $3L$-colouring of F using $\Delta + 2$ colours. In this colouring, clearly a maximum of $\Delta - 2$ colours are used to colour the vertices adjacent to v. We can use the remaining four colours including the colour given to v, to colour the other vertices in B, and obtain a $3L$-colouring of C with $\Delta + 2$ colours. If B is K_2, a similar argument holds.

The result (9) and Proposition 11 give :

Corollary 11.1 — For a K_3-free cactus C with $\Delta \geq 3$,

$$1 + \Delta \leq \chi_{3L} \leq 2 + \Delta.$$

Note that the lower bound is attained when C is a tree, and the upper bound is attained when C is C_n, $n \geq 4$.

Proposition 12 — For any K_3-free graph with $\Delta \geq 3$,

(i) $\chi_{2L} + 1 \leq \chi_{3L}$ ... (11)

(ii) $\chi_L \leq \left\lceil \frac{\chi_{3L}}{3} \right\rceil$ ... (12)

PROOF : (i) Let $\{V_1, V_2, ..., V_k\}$ be a χ_{3L}-partition of $V(G)$. Consider the partition $\{V_1 \bigcup V_2, V_3, ..., V_k\}$ of $V(G)$. Clearly, this is a $2L$-partition of $V(G)$. Hence $\chi_{2L} \leq k - 1 = \chi_{3L} - 1$.

(ii) Since the union of any three sets in a χ_{3L}-partition of $V(G)$ induces a linear forest, (12) follows.

We observe that if the minimum degree of G, $\delta(G) \geq 3$, then each set in any χ_{3L}-partition of $V(G)$ is independent.

Proposition 13 — For any K_3-free graph G,

$$\left\lceil \frac{\chi(G)}{2} \right\rceil \leq \left\lceil \frac{\chi_{3L}(G)}{3} \right\rceil$$

PROOF : The perfect chromatic number of a graph G, χ (G : perfect) is defined to be the least nonnegative integer k for which G has a perfect k-colouring (i.e. partition of the vertex set $V(G)$ into sets such that each set induces a perfect graph). For a triangle-free graph G, $\chi(G : \text{perfect}) = \left\lceil \frac{\chi(G)}{2} \right\rceil$ (see Brown and Corneil[1]). Consider a χ_{3L}-partition P of $V(G)$. The union of any three sets in P induces a linear forest. Linear forests being bipartite, are perfect graphs. Therefore, we have

$$\left\lceil \frac{\chi_{3L}(G)}{3} \right\rceil \geq \chi\ (G : \text{perfect}) = \left\lceil \frac{\chi(G)}{2} \right\rceil.$$

We now investigate graphs G with $\chi_{3L}(G) = |V(G)|$.

Proposition 14 — Let G be a connected K_3-free graph of order p with diameter 2, and $\delta(G) \geq 3$. Then, $\chi_{3L}(G) = p$.

PROOF : Let $u, v \in V(G)$. Suppose u and v are adjacent. Clearly, they cannot have the same colour in a $3L$-colouring of G. For otherwise, since $d(u) \geq 3$, we can find three sets V_i, V_j and V_k in a χ_{3L}-partition of $V(G)$ such that u and its neighbours induce $K_{1,3}$ in the subgraph $\langle V_i \bigcup V_j \bigcup V_k \rangle$. If u is not adjacent to v, then since diam $G = 2$, there exists a vertex w adjacent to both u and v. Since $d(w) \geq 3$, u and v cannot have the same colour in a $3L$-colouring of G, for otherwise, w, u, v and one more neighbour of w, say z induce $K_{1,3}$ in some subgraph $\langle V_i \bigcup V_j \bigcup V_k \rangle$, where V_i, V_j and V_k are sets in a χ_{3L}-partition of $V(G)$. Thus, no two vertices in G can have the same colour in any $3L$-colouring of $V(G)$, and $\chi_{3L}(G) = p$.

For example, let G be the graph obtained from C_8 by joining diametrically opposite vertices. Then, diam $G = 2$, $\delta(G) = 3$ and G is K_3-free, with $\chi_{3L}(G) = 8$.

Also, for the Petersen graph, $\chi_{3L} = 10$.

About the converse of Proposition 14, we have :

Proposition 15 — Let G be a connected K_3-free graph of order p with $\chi_{3L}(G) = p$. Then diam $G = 2$.

PROOF : Let $V(G) = \{v_1, v_2, ..., v_p\}$. For any two vertices, say v_1, v_2, consider the partition of $V(G)$, $P = \{V_2, V_3, ..., V_p\}$, where $V_2 = \{v_1, v_2\}$, $V_i = \{v_i\}$, $3 \leq i \leq p$. Since $\chi_{3L}(G) = p$, P is not a χ_{3L}-partition. Hence for V_2 there exist sets V_i and V_j in P such that $\langle V_2 \bigcup V_i \bigcup V_j \rangle = \langle \{v_1, v_2, v_i, v_j\} \rangle$ is not a linear forest. Since G is K_3-free, these four vertices either induce C_4 or $K_{1,3}$. In either case, $d(v_1, v_2) \leq 2$. This implies diam $G \leq 2$. Clearly, diam $G \neq 1$, for otherwise $G = K_p$, which is not true. Hence diam $G = 2$.

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