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ONE-POINT EXTENSIONS OF LOCALLY ALMOST PARALINDELÖF SPACES

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In this paper we give an example of almost paralindelöf space which is neither HP-closed nor paralindelöf. Further the one-point extensions of a locally almost paralindelöf space X are studied in some detail and it is proved that there is one such extension that admits an almost continuous function into every such extension leaving X pointwise fixed. It is also proved that a locally almost paralindelöf space belonging to a particular category admits one such extension that is a projective maximum in the class of all such extensions.

In our earlier paper[1] with T. G. Raghavan, we discussed several properties of almost paralindelöf spaces in relation to their corresponding minimal spaces. In this paper we study in some detail the one-point extensions of a locally almost paralindelöf space. We notice that in the class of all one-point almost paralindelöf extensions of a locally almost paralindelöf space X, there is one that admits an almost continuous function into every such extension leaving X pointwise fixed; in all such extensions of any space belonging to a particular class of locally almost paralindelöf spaces, there is a projective maximum.

All spaces discussed in this paper are HP-spaces, that is, Hausdorff spaces in which each G_δ-set is open. 'nbd' stands for neighbourhood. The letter N represents the set of natural numbers. □ marks the end of proof.

1. AN EXAMPLE AND PRELIMINARY RESULTS

Definition 1.1 — A topological space X is called almost paralindelöf if every open cover of X has an open locally countable refinement (not necessarily a cover) whose union is dense in X.

Example 1.2 — Let $A = [0, \Omega[$. Let $X = \{a\} \times A \times A \times A \cup \{b\} \times A \times A \times A \cup A \times A \cup \{a\} \times A \cup \{b\} \times A$. Let us define some subsets of X as follows :

$$B_a(P, \alpha, \beta) = \{a\} \times \{\alpha\} \times (A - P) \times \{\beta\}$$

$$B_b(Q,\alpha,\beta) = \{b\} \times \{\alpha\} \times (A-Q) \times \{\beta\}$$

$$V(P,Q,\alpha,\beta) = \{(\alpha,\beta)\} \bigcup B_a(P,\alpha,\beta) \bigcup B_b(Q,\alpha,\beta)$$

$$V(P, a, \beta) = \{(a, \beta)\} \bigcup \{a\} \times (A-P) \times A \times \{\beta\}$$

$$V(Q, b, \beta) = \{(b,\beta)\} \bigcup \{b\} \times (A-Q) \times A \times \{\beta\}.$$

Let us define a topology τ on X with the help of the following basic nbds. If $x \in (\{a\} \times A \times A \times A) \bigcup (\{b\} \times A \times A \times A)$, then $\{x\}$ is τ-open. If $x = (\alpha,\beta) \in X$, then a basic τ-open nbd of x is $V(P, Q, \alpha, \beta)$ for some countable subsets P and Q $(\subset A)$. If $x = (a, \beta) \in X$, then a basic τ-open nbd of x is $V(P, a, \beta)$ for some countable subset $P \subset A$. Similarly if $x = (b, \beta) \in X$, then a basic τ-open nbd of x is $V(Q, b, \beta)$ for some countable subset $Q \subset A$. Thus what results is an HP-space.

Claim (i) : (X, τ) is not HP-closed.

PROOF OF CLAIM (i) : For a general study of minimal-HP and HP-closed spaces, one may refer to Raghavan and Reilly[4]. Let

$$Z(a, \alpha) = \{a\} \times A \times A \times \{\alpha\}$$

$$Z(b, \alpha) = \{b\} \times A \times A \times \{\alpha\}$$

$$Z(\alpha) = (A \times \{\alpha\}) \cup \{(a, \alpha)\} \cup \{(b, \alpha)\}$$

$$U_\alpha = Z(a, \alpha) \cup Z(b, \alpha) \cup Z(\alpha).$$

Let $x \in U_\alpha$. If $x = \{a, \xi, \eta, \alpha)$, then $\{(a, \xi, \eta, \alpha)$ is open. If $x = (b, \xi, \eta, \alpha)$, then $\{(b, \xi, \eta, \alpha)\}$ is open. If $x = (\xi, \alpha)$, then $x \in V(P, Q, \xi, \alpha) \subset Z(a, \alpha) \cup Z(b, \alpha) \subset U_\alpha$. If $x = (a, \alpha)$, then $x \in V(P, a, \alpha) \subset Z(a, \alpha) \subset U_\alpha$. If $x = (b, \alpha)$, then $x \in V(Q, b, \alpha) \subset Z(b, \alpha) \subset U_\alpha$. Thus U_α is open for each $\alpha \in A$ and $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ is an open cover of X. If $x \in U_\beta$, then $x \notin U_\alpha$ for any $\alpha \in A$ with $\beta \neq \alpha$. Thus U_α is a closed and open subset of X. That is $U_\alpha \cap U_\beta = \phi$ if $\alpha \neq \beta$ and cl $U_\alpha = U_\alpha$. Thus $\mathcal{U}$ does not admit a countable subfamily $\mathcal{V}$ such that $X = \text{cl}\,(\bigcup \mathcal{V})$.

Claim (ii) : (X, τ) is almost paralindelöf.

PROOF OF CLAIM (ii) : Let $\mathcal{U}$ be any open cover of X. Let $\mathcal{U}_\alpha$ be the collection $\{U_\alpha \cap U \mid U \in \mathcal{U}\}$ where each U_α is as defined in the proof of claim (i). Thus $\mathcal{U}_\alpha$ is an open cover of U_α. Thus there exist sets $V(a, \alpha)$, $V(b, \alpha)$, both in $\mathcal{U}_\alpha$ such that

$$(a, \alpha) \in V(P_0, a, \alpha) \subset V(a, \alpha)$$

$$(b, \alpha) \in V(Q_0, b, \alpha) \subset V(b, \alpha)$$

for countable subsets P_0 and $Q_0 \subset A$. Let $R_0 = P_0 \bigcup Q_0$. Consider $(\gamma, \alpha) \in U_\alpha$. Then there is an open set $V(\gamma, \alpha) \in \mathcal{U}_\alpha$ such that $(\gamma, \alpha) \in V(\gamma, \alpha)$. Further there is a basic open set $V(P(\gamma), Q(\gamma), \gamma, \alpha)$ for some countable subsets $P(\gamma)$, $Q(\gamma)$ $(\subset A)$ such that $(\gamma, \alpha) \in V(P(\gamma), Q(\gamma), \gamma, \alpha) \subset V(\gamma, \alpha)$. Let $\mathcal{B}_\alpha = \{V(P(\gamma), Q(\gamma), \gamma, \alpha) \mid \gamma \in R_0\}$. If $\xi \in P(\gamma)$, then there exists an open nbd of (a, γ, ξ, α), say, $V(a, \gamma, \xi, \alpha) \in \mathcal{U}_\alpha$, since the element (a, γ, ξ, α) is not covered by $V(P, a, \alpha)$, $V(Q, b, \alpha)$ and the members of $\mathcal{B}_\alpha$. Thus $\{(a, \gamma, \xi, \alpha)\}$ is a basic open set such that $\{(a, \gamma, \xi, \alpha)\} \subset V(a, \gamma, \xi, \alpha)$. Thus we can form the collection $C_\alpha = \{(a, \gamma, \xi, \alpha)\} \mid \gamma \in P_0, \xi \in P(\gamma)\}$. If $\eta \in Q(\gamma)$, then there is an open set $V(b, \gamma, \xi, \alpha) \in \mathcal{U}_\alpha$ since the element (b, γ, ξ, α) is not covered by $V(P_0, a, \alpha)$, $V(Q_0, b, \alpha)$ and the members of $\mathcal{B}_\alpha$ and those of C_α. Thus $\{(b, \gamma, \xi, \alpha)\}$ is a basic open set such that $\{(b, \gamma, \xi, \alpha)\} \subset V(b, \gamma, \xi, \alpha)$. Thus we can form the collection $\mathcal{D}_\alpha = \{\{(b, \gamma, \xi, \alpha)\} \mid \gamma \in Q_0, \eta \in Q(\gamma)\}$. Let $\mathcal{V}_\alpha = \{V(P_0, a, \alpha)\} \bigcup \{V(Q_0, b, \alpha)\} \bigcup \mathcal{B}_\alpha \bigcup C_\alpha \bigcup \mathcal{D}_\alpha$. Then clearly each member of $\mathcal{V}_\alpha$ is contained in some member of $\mathcal{U}_\alpha$. Each member of $\mathcal{V}_\alpha$ is τ-open. $\mathcal{V}_\alpha$ is countable. If $x \in U_\beta$ $(\beta \neq \alpha)$, then $U_\beta \bigcap V = \phi$ for each $V \in \mathcal{V}_\alpha$. In this way we have a locally countable collection $\mathcal{V} = \bigcup \{\mathcal{V}_\alpha \mid \alpha \in A\}$ which is a refinement of $\mathcal{U}$. Let us write $Z_\alpha = \bigcup \{V \mid V \in \mathcal{V}_\alpha\}$ and $Z = \bigcup \{Z_\alpha \mid \alpha \in A\} = \bigcup \{V \mid V \in \mathcal{V}\}$. If $x \in U_\alpha - Z_\alpha$ for some $\alpha \in A$, then each τ-open nbd of x meets Z_α so that each τ-open nbd of x meets Z and hence cl $Z = X$.

Claim (iii) : (X, τ) is not paralindelöf.

PROOF OF CLAIM (iii) : Every closed and open subspace of paralindelöf space is paralindelöf. Let us take the closed and open subset $U_\alpha (\subset X)$. Let

$$\mathcal{E}_\alpha = \{a, \xi, \eta, \alpha) \mid \xi, \eta \in A\}$$

$$\mathcal{F}_\alpha = \{(b, \xi, \eta, \alpha) \mid \xi, \eta \in A\}$$

$$\mathcal{G}_\alpha = \{V(P, Q, \zeta, \alpha) \mid P \text{ and } Q \text{ are countable subsets of } A, \zeta \in A\}$$

$$\mathcal{H}_\alpha = \{V(P, a, \alpha) \mid P \text{ is a countable subset of } A\}$$

$$\mathcal{K}_\alpha = \{V(Q, b, \alpha) \mid Q \text{ is a countable subset of } A\}.$$

Let $\mathcal{W} = \mathcal{E}_\alpha \cup \mathcal{F}_\alpha \bigcup \mathcal{G}_\alpha \bigcup \mathcal{H}_\alpha \bigcup \mathcal{K}_\alpha$. $\mathcal{W}$ is an open cover of U_α. It has no refinement which is locally countable. Indeed any open nbd of (a, α) will meet uncountably many members of those sets (in the refinement) $\subset$ members of $\mathcal{G}_\alpha$. Thus the example is complete.

Definition 1.3 — Let A be a subset in the space (X, τ). Let τ/A be the relative topology of τ on A.

(i) A is called almost paralindelöf if $(A, \tau/A)$ is almost paralindelöf.

(ii) A is called an APL-set in X if every τ-open cover $\mathcal{U}$ of A admits a τ-open refinement $\mathcal{V}$ (not necessarily a cover of A) which is locally countable and $A \subset$

cl $(\bigcup \{V \mid V \in \mathcal{V}\})$.

Proposition 1.4 — If $A_i\,(i \in N)$ is a countable family of subsets of X each of which is an APL-set in X, then $\bigcup \{A_i \mid i \in N\}$ is an APL-set in X. □

Proposition 1.5 — Let $A\ (\subset X)$ be regular closed such that A = cl F for some open subset $F(\subset X)$. Then the following are equivalent :

(i) A is an almost paralindelöf subset of X.

(ii) A is an APL-set in X.

PROOF : (i) $\Rightarrow$ (ii). Let $\mathcal{U}$ be an open cover of A in X. Then $\mathcal{U}_A = \{U \cap A \mid U \in \mathcal{U}\}$ is a relative open cover of A. $\mathcal{U}_A$ has a refinement $\mathcal{V}_A$ each member of which is relatively open and $\mathcal{V}_A$ is locally countable and the closure of the union of the members of $\mathcal{V}_A$ is A. If $V \in \mathcal{V}_A$ there exists $U \in \mathcal{U}$ such that $V \subset U \cap A$. There exists an open set U' such that $U' \cap A = V \subset U \cap A$. Since A = cl F, $U' \cap F \neq \phi \neq U \cap F$ and $U' \cap F \subset U \cap F$. Also $\mathrm{cl}(U' \cap F) = \mathrm{cl}(U' \cap A)$ and $\sigma\text{-cl}(U' \cap A) \subset \mathrm{cl}(U' \cap A) = \mathrm{cl}(U' \cap F)$ where $\sigma = \tau/A$. Thus $\mathcal{V} = \{V \cap F \mid V \in \mathcal{V}_A\}$ is an open refinement of $\mathcal{U}$ and $\mathcal{V}$ is locally countable. Further the closure of the union of $\mathcal{V}$ covers A.

(ii) $\Rightarrow$ (i). Suppose $\mathcal{U}_A$ be a relatively open cover of A. If $U' \in \mathcal{U}_A$ then $U' = U \cap A$ for some $U \in \tau$. Consider $\mathcal{U} = \{U \mid \mathcal{U} \in \tau$ and $U \cap A = U'$ for some $U' \in \mathcal{U}_A\}$. Then $\mathcal{U}$ is an open cover of A; and it admits a locally countable open refinement $\mathcal{V}$ such that $\mathrm{cl}(\bigcup \mathcal{V}) \supset A$. Thus int $A \subset A$ = cl int $A \subset \bigcup \{\mathrm{cl}\ V \mid V \in \mathcal{V}\}$. If $x \in$ int A, then $x \in$ cl V for some $V \in \mathcal{V}$ so that if P is an open nbd of x, $P \cap V \neq \phi$. Indeed $P \cap V \cap \mathrm{int}\ A \neq \phi$ so that $x \in \mathrm{cl}\ (V \cap \mathrm{int}\ A)$. Thus int $A \subset \bigcup \{\mathrm{cl}(V \cap \mathrm{int}\ A) \mid V \in \mathcal{V}\} = \bigcup \{\mathrm{cl}(V \cap A) \mid V \in \mathcal{V}\}$. Hence $A \subset \mathrm{cl}\ (\bigcup \{V \cap A \mid V \in \mathcal{V}\})$ so that $A = \sigma\text{-cl}(\bigcup \{V \cap A \mid V \in \mathcal{V}\})$. Hence $\mathcal{V}_A = \{V \cap A \mid V \in \mathcal{V}\}$ is a σ-open refinement of $\mathcal{U}_A$ and it is locally countable such that the closure of the union of $\mathcal{V}_A$ is A.

Proposition 1.6 — If X is almost paralindelöf, A regular closed and A = cl F for some open subset $F\ (\subset X)$, then A is an APL-set in X or equivalently almost paralindelöf.

PROOF : Let $\mathcal{U}$ be an open cover of cl F. If $\mathcal{U}$ covers X, $\mathcal{U}$ admits an open refinement $\mathcal{V}$ which is locally countable and $X = \mathrm{cl}(\bigcup \mathcal{V}) \supset A$. If $\mathcal{U}$ does not cover X, then $\mathcal{U}' = \mathcal{U} \cup \{X - A\}$ is an open cover of X. Then $\mathcal{U}'$ has an open refinement $\mathcal{V}''$ which is locally countable and $X = \mathrm{cl}(\bigcup \mathcal{V}'')$. Let $\mathcal{V}_1$ be all those sets in $\mathcal{V}''$ such that $\mathcal{V} \in \mathcal{V}_1$ implies $V \subset X - \mathrm{cl}\ F$. Then $\bigcup \{\mathrm{cl}\ V \mid V \in \mathcal{V}_1\} =$

$\text{cl}(\bigcup\{V \mid V \in \mathcal{V}_1\}) \subset X - \text{int cl } F$. Thus $\bigcup\{\text{cl } V \mid V \in \mathcal{V}' - \mathcal{V}_1\} = \text{cl}(\bigcup\{V \mid V \in \mathcal{V}' - \mathcal{V}_1\}) \subset \text{int cl } F$ and hence covers cl F. □

Propositions 1.4, 1.5 and 1.6 imply the following result.

Proposition 1.7 — If F_i $(i \in N)$ are open such that cl F_i $(i \in N)$ are almost paralindelöf, then $\bigcup\{\text{cl } F_i \mid i \in N\}$ is also almost paralindelöf. □

An immediate consequence of Proposition 1.6 is the next statement.

Proposition 1.8 — The following are equivalent for a space X :

(i) X is almost paralindelöf.

(ii) The closure of every open subset of X is an APL-set in X.

(iii) The closure of every open subset of X is almost paralindelöf. □

Proposition 1.9 — If B is a regular closed subset of X and $B \subset A$ which is an APL-set in X, then B is an APL-set in X.

PROOF : The proof of this is similar to that of Proposition 1.6. □

Proposition 1.10 — If $B \subset A \subset X, A$ is open and B is an APL-set in X, then B is APL-set in A.

PROOF : Let τ be the topology on X. Let $\mathcal{U}_A$ be a σ-open cover of B where $\sigma = \tau/A$. Since A is τ-open, $\mathcal{U}_A$ is a τ-open cover of B. Then there exists a τ-open refinement $\mathcal{V}_A$ such that $\mathcal{V}_A$ is locally countable and $B \subset \tau\text{-cl}(\bigcup\{V \mid V \in \mathcal{V}_A\}) = \bigcup\{\tau\text{-cl } V \mid V \in \mathcal{V}_A\}$. Thus $B = B \cap A \subset \bigcup\{\tau\text{-cl } V \cap A \mid V \in \mathcal{V}_A\} = \bigcup\{\sigma\text{-cl } V \mid V \in \mathcal{V}_A\}$. Hence B is an APL-set in A. □

2. ONE-POINT EXTENSIONS

In this section we describe the one-point almost paralindelöf extensions of locally almost paralindelöf spaces.

Definition 2.1 — A topological space (X, τ) is locally almost paralindelöf if each point of X has an open nbd whose closure is an APL-set in X.

A topological space (Y, σ) is an extension of (X, τ) if $X \subset Y$, $\tau = \sigma/X$ and σ-cl $X = Y$. An HP-space (Y, σ) is said to be a one-point almost paralindelöf extension of (X, τ) if (Y, σ) is an extension of (X, τ), (Y, σ) is almost paralindelöf and $Y - X$ is a singleton.

Let (Y, σ) and (Z, ρ) be extensions of a topological space (X, τ). Then (Y, σ) is said to be projectively larger than (Z, ρ) or (Z, ρ) is said to be projectively smaller than (Y, σ), if there is a continuous surjection from Y to Z, which leaves X pointwise fixed. (Y, σ) and (Z, ρ) are said to be isomorphic if there exists a homeomorphism which leaves X pointwise fixed. Let $\mathcal{K}$ be a class of almost paralindelöf extensions of (X, τ). If $(Y, \sigma) \in \mathcal{K}$, (Y, σ) is called a projective maximum (minimum) in $\mathcal{K}$ if (Y, σ) is projectively larger (smaller) than (Z, ρ) for every $(Z, \rho) \in \mathcal{K}$.

Proposition 2.2 — Let (Y, σ) be a one-point almost paralindelöf extension of (X, τ). Then

(i) X is open in Y.

(ii) X is almost paralindelöf if $Y - X$ is open.

(iii) X is locally almost paralindelöf.

PROOF : $Y - X$ is a singleton and hence closed in Y so that X is open. If $Y - X$ is open then X is both open and closed so that it is almost paralindelöf by Proposition 1.6. Further if $x \in X$ there exists a σ-open nbd U of x such that $\pi \notin$ σ-cl U where $\{\pi\} = Y - X$. Further σ-cl U is an APL-set in Y. Since $\sigma\text{-cl } U \cap X = \tau\text{-cl } U = \sigma\text{-cl } U$ and X is open in Y, it follows from Proposition 1.10 that τ-cl U is an APL-set in X. Thus X is locally almost paralindelöf. □

A filter (base) $\mathcal{F}$ in (X, τ) is a σ-filter (base) if the filter (generated by the filter base) is closed under countable intersection. A σ-filter $\mathcal{F}$ is said to be an open σ-filter if $\mathcal{F} \subset \tau$.

The concept of almost continuous function was defined and studied by Singal and Singal[5]. A function $f : X \to Y$ is called almost continuous if for each $x \in X$ and each regular open nbd V of $f(x)$, there is an open nbd U of x such that $f(U) \subset V$ or equivalently the inverse image of every regular open subset of Y is open in X.

Theorem 2.3 — Let (X, τ) be a locally almost paralindelöf space which is not almost paralindelöf. Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$. Then

(i) $\hat{\tau} = \tau \cup \{\{\pi\} \cup U \mid U \in \mathcal{F}\}$ is a Hausdorff P topology on X^* where $\mathcal{F}$ is the open σ-filter generated by $\{V \mid V \in \tau$ and $X - V$ is an APL-set in $X\}$,

(ii) $(X^*, \hat{\tau})$ is a one-point almost paralindelöf extension of (X, τ).

PROOF : Let $\mathcal{G} = \{U \mid U \in \tau$ and $X - U$ is an APL-set in $X\}$. Since X is not almost paralindelöf or equivalently an APL-set in itself, $\phi \notin \mathcal{G}$. If $U_i \in \mathcal{G}$ $(i \in N)$, then $X - U_i$ is an APL-set in X for each $i \in N$. Thus $\bigcup \{X - U_i \,(i \in N)\}$ is an APL-set in X so that $\bigcap \{U_i \mid i \in N\} \in \mathcal{G}$. Hence $\mathcal{G}$ is an open σ-filter base. It is easily seen now $(X^*, \hat{\tau})$ is an HP-space.

Let $\mathcal{U}$ be a $\hat{\tau}$-open cover of X^*. There exists a $\hat{\tau}$-open set $U_0 \in \mathcal{U}$ such that $\pi \in U_0$. Further there exists a τ-open set V_0 such that $\pi \in \{\pi\} \cup V_0 \subset U_0$ and that $X - V_0$ is an APL-set in X. Let $\mathcal{U}_0 = \{U \cap X \mid U \in \mathcal{U}\}$. Then $\mathcal{U}_0$ is a τ-open cover of $X - V_0$ so that $\mathcal{U}_0$ admits a τ-open refinement $\mathcal{V}_0$ such that $\tau\text{-cl}(\bigcup \mathcal{V}_0) \supset X - V_0$ and $\mathcal{V}_0$ is locally countable. Let $\mathcal{V}_1 = \{V - \tau\text{-cl } V_0 \mid V \in \mathcal{V}_0\}$ and $\mathcal{V} = \mathcal{V}_1 \cup \{V_0\}$. Then $\mathcal{V}$ is a $\hat{\tau}$-open refinement of $\mathcal{U}$. Further $\mathcal{V}$ is locally countable. Let $Z = \hat{\tau}\text{-cl}\,(\bigcup \{V - \tau\text{-cl } V_0 \mid V \in \mathcal{V}_0\}) \cup \hat{\tau}\text{-cl } V_0$. Suppose $x \in X^*$ and $x \notin \hat{\tau}\text{-cl } V_0$. Then, since $x \neq \pi$, $x \notin \hat{\tau}$-cl V_0 so that $x \in X - V_0$. Hence $x \in \tau$-cl V for some $V \in \mathcal{V}_0$ so that $x \in \hat{\tau}\text{-cl } (V - \tau\text{-cl } V_0)$. Hence $Z = X^*$. Thus the space $(X^*, \hat{\tau})$ is almost paralindelöf.

It is known that in a P-space the intersection of countably many regular open sets is regular open. One may refer to the first part of the proof of Theorem 2.4 of Raghavan and Reilly[3]. It was also observed by them that if X is a P-space, then X_s is also a P-space where X_s is the semiregularization of X. One can also refer to Theorem 2 and Corollary 2 of Mrsevic *et al.*[2].

Theorem 2.4 — Let (X, τ) be a locally almost paralindelöf space which is not almost paralindelöf. Let $X^* = X \cup \{\pi\}$ where $\pi \notin X$. Then

(i) if $\mathcal{G}$ is an open σ-filter generated by $\mathcal{F} = \{U : U$ is regular open and $X - U$ is an APL-set in $X\}$. $\tau^+ = \tau \cup \{\{\pi\} \cup G \mid G \in \mathcal{G}\}$ is a Hausdorff P-topology on X^*,

(ii) (X^*, τ^+) is a one-point almost paralindelöf extension of (X, τ),

(iii) if (Y, σ) is any other one-point almost paralindelöf extension of (X, τ), then there is an almost continuous $f : X^* \to Y$ which leaves X pointwise fixed.

PROOF : Since X is not almost paralindelöf, $\phi \notin \mathcal{G}$. Further if $X - U_i\,(i \in N)$ are APL-sets in X, $\bigcup \{X - U_i \mid i \in N\} = X - \bigcap U_i \mid \{i \in N\}$ is an APL-set in X. Since the intersection of countably many regular open sets in P-space is regular open, $\bigcap \{U_i \mid i \in N\}$ is a regular open and a member of $\mathcal{F}$. Thus $\mathcal{F}$ is a regular open σ-filter base. Now it is easily seen that (X^*, τ^+) is an HP-space.

The arguments to prove $(X^*, \hat{\tau})$ is almost paralindelöf in Theorem 2.3 carry over to this case also.

Let $Y = X \cup \{\eta\}$. Define $f : X^* \to Y$ by taking $f(x) = x$ for all $x \in X$ and $f(\pi) = \eta$. It is enough to verify the almost continuity of f at π. Let U be regular open nbd of η in (Y, σ). Then $f^{-1}(U) = (U \cap X) \cup \{\pi\}$. Since Y is almost paralindelöf, $Y - U \subset X$ and X is open, $Y - U$ is an APL-set in X, by Proposition 1.10. Since $Y - U = X - U \cap X$, $(U \cap X) \cup \{\pi\} \in \tau^+$. □

Let (X, τ) be a space. The following is called condition (#) for this paper :

For any countable collection $\{U_i \mid i \in N\} \subset \tau$

(#)$\bigcap$ {int cl $U_i \mid i \in N$ = int cl $(\bigcap \{U_i \mid i \in N\})$.

There are examples of spaces in which the condition (#) is satisfied.

Example 2.5 — (i) The discrete space always satisfies the condition (#).

(ii) Suppose X is a space such that every open subset of X is closed (In particular, if X is Hausdorff further, X is discrete). Then X satisfies the condition (#).

(iii) Let X be the set of reals endowed with the countable complement topology. Then X satisfies (#).

Example 2.6 — Let $X = [1, \Omega[$ be endowed with the discrete topology. Let $Y = X \cup \{\Omega\}$ be a one-point extension such that U is an open nbd of Ω in Y if $Y - U$ is countable. Y satisfies the condition (#).

Example 2.7 — This is a wellknown example of H-closed Urysohn space. Consider $X = \{\mathbf{a}, \mathbf{a}_{ij}, c_i \mid i, j \in N\}$. A basic open nbd of $\mathbf{a}$ is of the form $\{\mathbf{a}\} \bigcup \{\mathbf{a}_{ij} \mid i \geq k, j \in N\}$ for some integer $k \geq 1$. A basic open nbd of c_i is of the form $\{c_i\} \bigcup \{\mathbf{a}_{ij} \mid j \geq k\}$ for some integer $k \geq 1$. A basic open nbd of $\mathbf{a}_{ij}$ is of the form $\{a_{ij}\}$. Let $U_k = \{a\} \bigcup \{a_{ij} \mid i \geq k, \ j \in N\}$. Then clearly int cl $(\bigcap \{U_k \mid k \in N\}) = \phi$ while $\bigcap \{\text{int cl } U_k \mid k \in N\} = \{\mathbf{a}\}$. Thus (#) is not satisfied in this case.

Theorem 2.8 — Let (X, τ) be locally almost paralindelöf satisfying (#) but not almost paralindelöf. Let $X^* = X \bigcup \{\pi\}$ where $\pi \notin X$. Then

(i) $\tau^* = \tau \bigcup \{\{\pi\} \bigcup U \mid U \in \tau$ and $X -$ int cl U is an APL-set in $X\}$ is a Hausdorff P topology on X^*,

(ii) (X^*, τ^*) is a one-point almost paralindelöf extension of (X, τ),

(iii) (X^*, τ^*) is a projective maximum in the set of all one-point almost paralindelöf extensions of (X, τ).

PROOF : Let $\mathcal{G} = \{U \mid U \in \tau$ and $X -$ int cl U is an APL-set in $X\}$. Since X is not almost paralindelöf, $\phi \notin \mathcal{G}$. Further if $X -$ int cl U_i are APL-sets in $X (i \in N)$, then $\bigcup \{X -$ int cl $U_i \mid i \in N\}$ is an APL-set in X so that $X - \bigcap \{\text{int cl } U_i \mid i \in N\} = X -$ int cl $(\bigcap \{U_i \mid i \in N\})$ is an APL-set in X. Hence if $U_i \in \mathcal{G}$ $(i \in N)$, then $\bigcap \{U_i \mid i \in N\} \in \mathcal{G}$. Suppose $V \in \tau$ and $V \supset U$ for some $U \in \mathcal{G}$. Then $X -$ int cl $V \subset X -$ int cl U so that $X -$ int cl V is an APL-set in X by Proposition 1.9 and hence $V \in \mathcal{G}$. Thus $\mathcal{G}$ is an open σ-filter. It is easily seen that (X^*, τ^*) is an HP-space.

Let $\mathcal{U}$ be a τ^*-open cover of X^*. The arguments in the proof of Theorem 2.3 that $(X^*, \hat{\tau})$ is almost paralindelöf carry over to this case also, the only replacement being $X -$ int cl V_0 in the place of $X - V_0$.

Let (Y, σ) be any other one-point almost paralindelöf extension of (X, τ). Define $f : X^* \to Y$ by taking $f(x) = x$ for all $x \in X$ and $f(\pi) = \xi$ where $\{\xi\} = Y - X$. Since X is Hausdorff, $X \in \sigma$, $X \in \tau$ and f is identity on X, it is enough to verify the continuity of f at π. Let U be a σ-open nbd of ξ. Then $f^{-1}(U) = (U \cap X) \bigcup \{\pi\}$. Since Y is almost paralindelöf, σ-cl $(Y - \sigma\text{-cl } U) \subset X$ and X is open, σ-cl$(Y - \sigma$-cl $U)$ is an APL-set in X, by Proposition 1.10. Since σ-cl $(Y - \sigma$-cl $U) = X -$ int cl $(U \cap X)$, $\{\pi\} \bigcup (U \cap X) \in \tau^*$. Thus f is continuous. □

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