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SOME ABELIAN THEOREMS FOR THE DISTRIBUTIONAL KONTOROVICH-LEBEDEV TRANSFORMATION

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In this paper it is proposed to obtain some initial value and final value Abelian theorems for the distributional Kontorovich-Lebedev transformation, which is defined by

$$F(y) = \left\langle f(t), \frac{K_{iy}(t)}{\sqrt{t}} \right\rangle, \quad t, y > 0$$

where $K_{iy}(t)$ is the Macdonald function.

1. INTRODUCTION

The conventional Kontorovich-Lebedev transformation is defined by

$$F(y) = \int_0^\infty \frac{K_{iy}(t)}{\sqrt{t}} f(t)\, dt \qquad \text{... (1.1)}$$

where $K_{iy}(t)$ is Macdonald function (see Lebedev[1]), t and y are positive real numbers. The peculiarity of the transformation (1.1) lies in the fact that it involves integration with respect to parameter. Its extension to distributions of compact supports which involves some complicated analysis has been done by Zemanian[4]. The transformation (1.1) has also been extended to a larger space of distributions and the theory thus developed has been applied to a Dirichlet's problem for a wedge by Pathak and Pandey[3]. Recently, Pandey[2] has given some Abelian theorems for the conventional transformation (1.1). The peculiarity of the Abelian theorems lies in the fact that these theorems relate the behaviour of generating function $F(y)$ as $y \to \infty$ ($y \to \infty$) to the behaviour of determining function $f(t)$ as $t \to 0$ ($t \to \infty$). These results are also referred to as initial value and final value Abelian theorems. So the aim of the present paper is to establish some Abelian theorems for the distributional Kontorovich-Lebedev transformation defined by

$$F(y) = \langle f(t), \frac{K_{iy}(t)}{\sqrt{t}} \rangle, \quad t, y > 0. \qquad \dots (1.2)$$

In this paper we assign limits to certain regular distributions. This means that there exists a measurable function $g(t)$ that is Lebesgue integrable on every interval (a, b) for which $c < a < b < d$, and

$$\langle f, \varphi \rangle = \int_c^d g(t)\, \varphi(t)\, dt$$

for every smooth function φ defined on (c, d) whose support is a compact subset of (c, d). Also consider that one of the functions in this class (say, $g(t)$ again) possesses the limit, $\lim_{t \to c+} g(t) = \lambda$. We may assign the unique limit λ to f as $t \to c+$ when at least one function in the said class possesses this limit and we shall write

$$\lim_{t \to c+} f(t) = \lambda.$$

2. Notations and Terminology

In this work we follow the notations and terminology given in Pathak and Pandey[3].

Let $I = (0, \infty)$, $t \in I$ and α be a fixed positive real number. Let $\xi(t)$ be an infinitely differentiable function defined over I, satisfying $\xi(t) > 0$ for all $t > 0$ and such that

$$\xi(t) = \sqrt{t}, \quad 0 < t \le 1$$
$$= t^{(3/2) - \alpha} \log t, \quad t > 2.$$

Then the space $K_\alpha(I)$ is defined to be the collection of all infinitely differentiable complex-valued functions $\varphi(t)$ on I with the property

$$\gamma_k(\varphi) = \sup_{0 < t < \infty} \left| \xi(t)\, \Delta_t^k\, \varphi(t) \right| < \infty$$

for each $k = 0, 1, 2, \dots$ where $\Delta_t = t^2 D^2 + 2tD - t^2$, $D = \frac{d}{dt}$. The topology over the space $K_\alpha(I)$ is generated by the separating collection of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ [see Zemanian[5], p. 8]. The space $K_\alpha(I)$ is a locally convex, sequentially complete, Hausdorff topological vector space. The dual space of $K_\alpha(I)$ is denoted by $K_\alpha'(I)$. For $f \in K_\alpha'(I)$ we define

$$F(y) = \langle f(y), \frac{K_{iy}(t)}{\sqrt{t}} \rangle \qquad \dots (2.1)$$

for fixed $y > 0$, since $\frac{K_{iy}(t)}{\sqrt{t}} \in K_\alpha(I)$ in view of the result

$$\Delta_t\left[\frac{K_{iy}(t)}{\sqrt{t}}\right] = -\left(\frac{1}{4} + y^2\right)\frac{K_{iy}(t)}{\sqrt{t}}.$$

Now we assume that $f \in K_\alpha'(I)$ and $\varphi \in K_\alpha(I)$. Also assume that $\langle f, \varphi \rangle = 0$ whenever $\varphi(t) = 0$ on $T \leq t < \infty$ $(T > 0)$. In this case the support of f is a subset of $[T, \infty]$. Since f is the zero distribution on $(0, T)$ by above convention $\lim_{t \to 0+} f(t) = 0$. Similarly assume that $\langle f, \varphi \rangle = 0$ whenever $\varphi(t) = 0$ on $0 < t \leq T$ $(T < \infty)$. In this case support of f is contained in $[0, T]$, and by the same convention we write

$$\lim_{t \to \infty} f(t) = 0.$$

3. Abelian Theorems for Conventional Kontorovich-Lebedev Transformation

We use the Abelian theorems for classical Kontorovich-Lebedev transformation proved by Pandey[2]. They are given as under :

Theorem 3.1 [*An initial-value theorem for* (1.1)] — Let Re $\eta < \frac{1}{2}$. Let $f(x)$ be a measurable function on the interval $(0, \infty)$ such that $x^{-\frac{1}{2}} f(x)$ is Lebesgue integrable on every interval of the form (X, ∞), $X > 0$.

Assume that

$$\lim_{x \to 0+} x^\eta f(x) = \lambda \qquad \text{... (3.1)}$$

where λ is a complex number and $F(y)$, the Kontorovich-Lebedev transformation defined by (1.1). Then

$$\lim_{y \to \infty} [F(y) - \lambda G(y, \eta)] = 0$$

where

$$G(y, \eta) = 2^{-\eta-(3/2)}\left[\Gamma\left(\frac{1-2\eta}{4}\right)\right]^2 \times \left\{\prod_{n=0}^{\infty}\left[1 + y^2\left(\frac{1-2\eta+4n}{2}\right)^{-2}\right]^{-1}\right\} \qquad \text{... (3.2)}$$

for Re $(\frac{1}{2} - \eta) > 0$.

Theorem 3.2 [*A final-value theorem for* (1.1)] — Let Re $\eta < 1/2$. Assume that $f(x)$ is a measurable in $(0, \infty)$ such that $x^{1/2} f(x)$ is Lebesgue integrable on every interval of the form $0 < x < X$ $(X < \infty)$ and that there exists a complex λ such that

$$\lim_{x \to \infty} x^{\eta} f(x) = \lambda. \qquad \text{... (3.3)}$$

Then with $F(y)$ and $G(y, n)$ as defined in (1.1) and (3.2) respectively,

$$\lim_{y \to \infty} [F(y) - \lambda\, G(y, \eta)] = 0. \qquad \text{... (3.4)}$$

We now obtain Abelian theorems for the distributional Kontorovich-Lebedev transformation defined by (2.1) i.e. we extend the above theorems to a certain class of distributions.

4. Abelian Theorems for the Distributional Kontorovich-Lebedev Transformation

First we extend Theorem 3.1 to certain distributions.

Theorem 4.1 [*An initial-value theorem for* (2.1)] — Assume that $f \in K_\alpha'(I)$ can be decomposed into $f = f_1 + f_2$ where f_1 is a classical function and $f_2 \in E'(I)$ is of order r.

Let Re $\eta < \frac{1}{2}\cdot$ Assume that f_1 satisfies the hypothesis of Theorem 3.1. If $F(y)$ is the distributional Kontorovich-Lebedev transformation of $f(t)$ and $G(y, \eta)$ be defined as in (2.1) and (3.2) respectively. Then

$$\lim_{y \to \infty} \left[F(y) - G(y, \eta) \lim_{t \to 0+} t^{\eta} f(t) \right] = 0.$$

Proof : From Pathak and Pandey[3] $F_2(y) = \langle f_2(t), \frac{K_{iy}(t)}{\sqrt{t}} \rangle$, $y > 0$ is a smooth function. By Theorem 2.3-1 of Zemanian[5] (p. 37), the support of f_2 is a compact subset of I. Choose $\psi(t) \in D(I)$ such that $\psi(t) \equiv 1$ on a neighbourhood of the support of f_2. Thus

$$F_2(y) = \langle f_2(t), \ \psi(t) \frac{K_{iy}(t)}{\sqrt{t}} \rangle \cdot$$

For $\alpha > 0$, $K_\alpha(I)$ is subspace of $E(I)$. Moreover it is dense in $E(I)$ because $K_\alpha(I)$ contains $D(I)$ and $D(I)$ is dense in $E(I)$. Therefore, by Theorem 1.8-2 (see Zemanian[5], p. 20) $E'(I)$ is a subspace of $K_\alpha'(I)$ for any $\alpha > 0$. Hence f_2 is a member of $K_\alpha'(I)$ and as in Pathak and Pandey[3] (p. 30) $\psi(t) \frac{K_{iy}(t)}{\sqrt{t}}$ is a member of $K_\alpha(I)$. Consequently by boundedness property of $f_2 \in K_\alpha'(I)$, there exist a positive constant c and a nonnegative integer r such that

$$| F_2(y) | \leq c \max_{0 \leq k \leq r} \sup_{0 < t < \infty} \left| \xi(t) \Delta_t^k \left[\psi(t) \cdot \frac{K_{iy}(t)}{\sqrt{t}} \right] \right| .$$

Using an inductive process to deduce

$$\Delta_t^k[\theta\varphi] = \sum_{j=0}^{2k}\left(a_{2k-j}t^{2k-j} + b_{2k+2-j}t^{2k+2-j} + c_j\sum_{p=1}^{k} d_p t^{2p}\right) \times \frac{d^{2k-j}}{dt^{2k-j}}[\theta\varphi]$$

$$= \sum_{j=0}^{2k}\left[\left(a_{2k-j}t^{2k-j} + b_{2k+2-j}t^{2k+2-j} + c_j\sum_{p=1}^{k} d_p t^{2p}\right) \times \sum_{\nu=0}^{2k-j}\binom{2k-j}{\nu} D_t^{2k-\nu-j}\,\theta\cdot D_t^{\nu}\varphi\right]$$

where $a_0 = 0$, $a_{2k} = 1$, $b_{2k+2} = b_{2k+1} = 0$, $c_j = 0$ for $j = 0, 1, 2, \ldots, 2k-1$ and a_{2k-j}, b_{2k+2-j}, $0 \le j \le 2k$ and d_p $(1 \le p \le k)$ are real constants depending on k and j, we have

$$|F_2(y)| \le c \max_{0\le k\le r}\sup_{0<t<\infty}\left|\sum_{j=0}^{2k}\left(a_{2k-j}t^{2k-j} + b_{2k+2-j}t^{2k+2-j} + c_j\sum_{p=1}^{k} d_p t^{2p}\right) \times \sum_{\nu=0}^{2k-j}\binom{2k-j}{\nu} D_t^{2k-\nu-j}(\psi(t))\cdot\xi(t)\,D_t^{\nu}\left[\frac{K_{iy}(t)}{\sqrt{t}}\right]\right|.$$

Again, using the integral representation,

$$K_{iy}(t) = \int_0^\infty e^{-t\cosh x}\cos yx\,dx$$

and the fact that

$$\frac{d}{dt}[K_{iy}(t)] = -\frac{1}{2}[K_{iy-1}(t) + K_{iy+1}(t)]$$

and

$$\frac{d}{dt}\left[\frac{K_{iy}(t)}{\sqrt{t}}\right] = -\frac{1}{2}\left[\frac{K_{iy-1}(t)}{\sqrt{t}} + \frac{K_{iy+1}(t)}{\sqrt{t}}\right] - \frac{1}{2t}\frac{K_{iy}(t)}{\sqrt{t}}.$$

We have

$$D_t^{\nu}\left[\frac{K_{iy}(t)}{\sqrt{t}}\right] = \sum_{p=-\nu}^{\nu}\left[\frac{a_p}{t^{\nu-s|p|}} + \frac{b_p}{t^{\nu-2-|p|}}\right]\frac{K_{iy+p}(t)}{\sqrt{t}}$$

where a_p, b_p are suitably chosen constants depending on ν and p.

Let the constant $C_{2k,\nu,j}$ be a bound on $D_t^{2k-\nu-j}\,\psi(t)$. Also for every integer p, we have $|K_{iy+p}(t)| \le C_1$ for $0 < y < 1$, $|K_{iy+p}(t)| \le C_1\, e^{-(1/2)y\pi}y^{-1/2}$ for $2 <$

$y < \infty$ and $K_{iy+p}(t) = \underline{O}(1)$ as $t \to 0$, $K_{iy+p}(t) = \underline{O}(t^{-1/2} e^{-(1/2)t})$ as $t \to \infty$.

Therefore by Zemanian[5] we have

$$| F_2(y) | \leq C^1 \sup_{0<t<\infty} \left| \xi(t) \frac{K_{iy+p}(t)}{\sqrt{t}} \right|$$

$$\leq C^{11} y^{-(1/2)} e^{-(1/2)y\pi} \text{ for } 2 < y < \infty$$

where C^{11} is sufficiently large real number.

We see that $F_2(y) \to 0$ as $y \to \infty$. ... (4.1)

Now the support of $f_2 \in E'(I)$ is a compact subset of $0 < t < \infty$; by our convention in section 2,

$$\lim_{t \to 0+} t^{\eta} f_2(t) = 0.$$

Therefore Theorem 4.1 follows directly from Theorem 3.1. The proof is complete.

We now prove a final value Abelian theorem for the distributional Konorovich-Lebedev transformation which is an easy consequence of Theorem 4.1.

Theorem 4.2 [*A final-value theorem for* (2.1)] — Let Re $\eta < \frac{1}{2}$ · Assume that $f = f_1 + f_2$ where f_1 is a classical function satisfying the hypothesis of Theorem 3.2, and $f_2 \in E'(I)$. If $F(y)$ is the distributional Kontorovich-Lebedev transformation of $f(t)$ and $G(y, \eta)$ be defined as in (2.1) and (3.2) respectively. Then

$$\lim_{y \to \infty} \left[F(y) - G(y, \eta) \lim_{t \to \infty} t^{\eta} f(t) \right] = 0.$$

PROOF : The procedure for the proof of the Theorem 4.2, is very similar up to the steps given by eqn. (4.1).

On the other hand support of $f_2 \in E'(I)$ is a compact subset of $(0, t)$; by our convention in section 2,

$$\lim_{t \to \infty} t^{\eta} f_2(t) = 0.$$

Therefore the proof of the Theorem 4.2 now follows directly from Theorem 3.2. The proof is complete.

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