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SOME ABELIAN THEOREMS FOR DISTRIBUTIONAL $\overline{H}$-FUNCTION TRANSFORMATION

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In this paper, we introduce a new distributional integral transformation whose kernel is the product of an exponential function and $\overline{H}$-function due to Inayat Hussain[4]. We also establish the initial value and final value abelian theorems for this distributional transformation. The results given earlier by Saxena[7] and Charmichael and Pathak[2] can be derived as special cases of the results, proved in this paper.

1. INTRODUCTION

The results obtained in the present paper are of general character and various integral transforms involving most of the important special functions, as kernels, follow as special cases of our transform. Among them, the transforms studied by Zemanian[8, 9], Misra[6], Saxena[7], Charmichael and Pathak[2] are worth-mentioning here.

This paper deals with a new distributional integral transformation associated with the product of an exponential function and $\overline{H}$-function. Some abelian theorems for this transformation are established which extend the theorems proved earlier by Saxena[7] and Charmichael and Pathak[2] etc.

2. THE $\overline{H}$-FUNCTION

The $\overline{H}$-function, introduced by Inayat Hussain[4] in terms of Mellin-Barnes type contour integral, is defined by

$$\overline{H}_{p,q}^{m,n}\left[z \,\middle|\, \begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, & (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, & (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Theta(s)\, z^s\, ds \quad \ldots (2.1)$$

where

$$\Theta(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j - B_j s) \prod_{j=1}^{n} \left\{\Gamma(1-\alpha_j + A_j s)\right\}^{a_j}}{\prod_{j=m+1}^{q} \left\{\Gamma(1-\beta_j + B_j s)\right\}^{b_j} \prod_{j=n+1}^{p} \Gamma(\alpha_j - A_j s)} \quad \ldots (2.2)$$

which contains fractional powers of some of the Γ-functions. Here z may be real or complex but is not equal to zero and an empty product is interpreted as unity; p, q, m and n are integers such that $1 \le m \le q$, $0 \le n \le p$; A_j ($j = 1, \ldots, p$), B_j ($j = 1, \ldots, q$) are complex numbers. The exponents a_j ($j = 1, \ldots, n$) and b_j ($j = m+1, \ldots, q$) can take non-integer values. When these exponents take integer values, the H-function reduces to the familiar H-function due to Fox[3].

A detailed account of the function is available from the monograph by Mathai and Saxena[5]. Analytic continuation and asymptotic expansions of the H-function have been discussed by Braaksma[1].

Also, due to Inayat Hussain[4], it follows that

$$\overline{H}_{p,q}^{m,n}[z] = O(|z|^{g}) \text{ for small } z \quad \ldots (2.3)$$

where

$$g = \min_{1 \le j \le m} \operatorname{Re}\left(\frac{\beta_j}{B_j}\right).$$

Again, due to Inayat Hussain[4], we have

$$\overline{H}_{p,q}^{m,n}[z] = O(|z|^{h}) \text{ for large } z \quad \ldots (2.4)$$

where

$$h = \max_{1 \le j \le n} \operatorname{Re}\left\{a_j\left(\frac{\alpha_j - 1}{A_j}\right)\right\}.$$

The following condition is also satisfied :

$$\Omega = \sum_{j=1}^{m} |B_j| + \sum_{j=1}^{n} |a_j A_j| - \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=n+1}^{q} |A_j| > 0 \quad \ldots (2.5)$$

and

$$|\arg(z)| < \frac{1}{2}\Omega\pi. \quad \ldots (2.6)$$

3. Distributional $\overline{H}$-Function Transform

Let $f(t)$ be a locally integrable function. Further let the following conditions be satisfied :

(i) $f(t) = 0$ for $-\infty < t < T$

(ii) there exists a real number u ... (A)

such that

$$\exp(-wut)\ \overline{H}_{p,q}^{m,n}\left[ut \,\middle|\, \begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right]$$

is absolutely integrable over $-\infty < t < \infty$. Here w is a given positive number.

Now, we define the distributional $\overline{H}$-function transform as an operator $\overline{\mathcal{H}}$ which assigns a function $F(s)$ of the complex variable s to each locally integrable function $f(t)$ satisfying conditions (A). This operation $\overline{\mathcal{H}}$ is defined by

$$\overline{\mathcal{H}}\,[f(t)] \overset{\Delta}{=} F(s) \overset{\Delta}{=} \int_{-\infty}^{\infty} \exp(-wst)\ \overline{H}_{p,q}^{m,n} \times \left[(st) \,\middle|\, \begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] f(t)\, dt \qquad \text{... (3.1)}$$

with the assumption that only sectors, having $\overline{H}_{p,q}^{m,n}(st) = O(e^{st})$ at infinity with $w > 1$ and $s > 0$ as well as the sectors having the asymptotic expansion of the $\overline{H}$-function of algebraic order at infinity with $(ws) > 0$, should be considered. The function $F(s)$ is called the $\overline{H}$-function transform of the function $f(t)$. With the first condition of (A), we can write (3.1) as

$$\overline{\mathcal{H}}\,[f(t)] \overset{\Delta}{=} F(s) \overset{\Delta}{=} \int_{-\infty}^{\infty} \exp(-wst)\ \overline{H}_{p,q}^{m,n} \times \left[(st) \,\middle|\, \begin{matrix}(\alpha_j, A_j; a_j)_{1,n}, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] f(t)\, dt \qquad \text{... (3.2)}$$

which is known as the right sided $\overline{H}$-function transform, since the lower limit in the above integral is finite.

4. An Initial Value Abelian Theorem for $\overline{H}$-Function Transform

Theorem 4.1 — Let η be fixed real number such that $\eta + g > -1$. Let $f(t)$ be a complex valued function satisfying the following conditions :

(i) $|\, t^{h+1} f(t)\,| < \infty$ for $t > y > 0$.

(ii) $f(t)/t^{\eta}$ is absolutely continuous on $0 \le t < \infty$. ... (A')

If there exists a complex number α such that

$$\lim_{t \to 0+} \frac{f(t)}{t^{\eta}} = \alpha \qquad \text{... (4.1)}$$

then the following result holds :

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} F(\sigma)}{\overline{H}(1/w)} = \alpha \qquad \ldots (4.2)$$

where

$$\overline{H}\left(\frac{1}{w}\right) = \overline{H}^{m,n+1}_{p+1,q}\left[\frac{1}{w}\,\middle|\,\begin{matrix}(-\eta, 1)\,(\alpha_j, A_j; a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] \qquad \ldots (4.3)$$

and

$$F(\sigma) = \int_0^\infty \exp(-w\sigma t)\, \overline{H}^{m,n}_{p,q}\left[\sigma t\,\middle|\,\begin{matrix}(\alpha_j, A_j; a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] f(t)\, dt. \qquad \ldots (4.4)$$

PROOF : For $\eta > -1$ and $\sigma > 0$, we have

$$\int_0^\infty t^\eta \exp(-w\sigma t)\, \overline{H}^{m,n}_{p,q}\left[\sigma t\,\middle|\,\begin{matrix}(\alpha_j, A_j; a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] dt$$

$$= \int_0^\infty t^\eta \exp(-w\sigma t) \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(s)\, (\sigma t)^s\, ds\, dt.$$

Changing the order of integration which is permissible under the conditions given in (A'), we obtain

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(s)\, \sigma^s\, ds \int_0^\infty t^{\eta+s} \exp(-w\sigma t)\, dt$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Theta(s)\, \sigma^s\, ds\, \frac{\Gamma(\eta + s + 1)}{(w\sigma)^{\eta+s+1}}.$$

Thus

$$\int_0^\infty t^\eta \exp(-w\sigma t)\, \overline{H}^{m,n}_{p,q}\left[\sigma t\,\middle|\,\begin{matrix}(\alpha_j, A_j; a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] dt$$

$$= \frac{1}{(w\sigma)^{\eta+1}} \overline{H}^{m,n+1}_{p+1,q}\left[\frac{1}{w}\,\middle|\,\begin{matrix}(-\eta, 1)\,(\alpha_j, A_j; a_j)_{1,n},\ (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m},\ (\beta_j, B_j; b_j)_{m+1,q}\end{matrix}\right] \qquad \ldots (4.5)$$

where $w > 0$, $\min_{1 \le j \le m} \mathrm{Re}\left(\frac{\beta_j}{B_j} + \eta\right) > 0$ and

$$|\arg(\sigma)| < \frac{1}{2}\Omega\pi \text{ [}\Omega\text{ is defined by (2.5)]}.$$

By using the result (4.5) and assuming that $\sigma > 0$, $y > 0$, we may write

$$\left| (w\sigma)^{\eta+1} F(\sigma) - \alpha \bar{H}(1/w) \right| = \left| (w\sigma)^{\eta+1} \int_0^\infty \exp(-w\sigma t) \times \bar{H}_{p,q}^{m,n}[\sigma t] f(t)\, dt - \alpha (w\sigma)^{\eta+1} \int_0^\infty t^\eta \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t]\, dt \right| \quad \ldots (4.6)$$

$$= \left| (w\sigma)^{\eta+1} \int_0^\infty \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t]\, t^\eta \left\{ \frac{f(t)}{t^\eta} - \alpha \right\} dt \right|$$

$$\le (w\sigma)^{\eta+1} \int_0^y \left| t^\eta \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t] \right| \left| \frac{f(t)}{t^\eta} - \alpha \right| dt + (w\sigma)^{\eta+1} \int_y^\infty \left| \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t] (f(t) - \alpha t^\eta) \right| dt$$

$$= I_1 + I_2 \quad \ldots (4.7)$$

where I_1 and I_2 are self-explanatory related integral expressions. Now,

$$I_1 \le w^{\eta+1} \int_0^y \left| (\sigma t)^\eta \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t] \right| d(\sigma t) \sup_{0<t<y} \left| \frac{f(t)}{t^\eta} - \alpha \right|$$

$$\le w^{\eta+1} \int_0^{\sigma y} \left| x^\eta \exp(-wx) \bar{H}_{p,q}^{m,n}[x] \right| dx \sup_{0<t<y} \left| \frac{f(t)}{t^\eta} - \alpha \right|$$

where $\sigma t = x$.

Since $\frac{f(t)}{t^\eta} \to \alpha$ as $t \to 0+$, for $\varepsilon > 0$, there exists a $\delta > 0$ such that $\left| \frac{f(t)}{t^\eta} - \alpha \right| < \varepsilon$ for all $0 < t < \delta$ and $y < \delta$.

Also, by (2.3), the integral I_1 is said to be convergent if $\eta + g > -1$, where g is defined in (2.3). Hence as I_1 is independent of σ, for a given $\varepsilon > 0$, we can choose y small enough such that $I_1 < \varepsilon/2$.

We now consider I_2.

$$I_2 \le (w\sigma)^{\eta+1} \int_y^\infty \left| \exp(-w\sigma t) \bar{H}_{p,q}^{m,n}[\sigma t] (f(t) - \alpha t^\eta) \right| dt$$

$$\le (w\sigma)^{\eta+1} \int_y^\infty e^{-w\sigma t} \left| \bar{H}_{p,q}^{m,n}(\sigma t) \right| |f(t)|\, dt + (w\sigma)^{\eta+1} \alpha \int_y^\infty t^\eta e^{-w\sigma t} \left| \bar{H}_{p,q}^{m,n}(\sigma t) \right| dt$$

$$\leq A\, w^{\eta+1}\, \sigma^{\eta+h+1} \int_y^\infty e^{-wt}\, t^{h+1}\, |\, f(t)\, |\, dt + A\, w^{\eta+1}\, \sigma^{\eta+h+1}$$

$$\times \int_y^\infty t^{\eta+h}\, e^{-wt}\, dt.$$

Choose $|\, t^{h+1} f(t)\, | < \infty$ for $t > y > 0$ and $\eta + h + 1 < 0$, then the integrals are convergent and R.H.S. $\to 0$ as $\sigma \to \infty$. Here we have replaced σ by 1 within the integral signs as $\sigma \to \infty$.

Here $\left|\, \exp(-w\sigma t)\, t^{\eta}\, \bar{H}_{p,q}^{m,n}\, [\sigma t] \right|$ is bounded on $0 < t < \infty$, provided $\eta + g > -1$ where g is defined in (2.3). Hence, after fixing y in such a way that $I_1 < \varepsilon/2$, the integral I_2 can be made less than $\varepsilon/2$ by choosing σ large enough. This coupled with the argument on I_1 completes the proof.

In order to extend this result to distributions, we shall need the following.

Lemma — Let $f(t)$ be a $\bar{H}$-transformable distribution with its support in $y \leq t < \infty$, where $y > 0$. Also, let the real number c be such that $\exp(-wct)\, f(t)$ is in S', where S' is the space of distributions of slow growth defined by Zemanian[8] (p. 103). Then for $c + 1 < \sigma < \infty$,

$$|\, F(\sigma)\, | \leq M\, \sigma^{r} \exp[-w(\sigma - c)\tau] \qquad \ldots (4.8)$$

where M is a constant and τ is any real number satisfying $0 < \tau < y$.

PROOF : For $\sigma > c$, we have

$$F(\sigma) = \langle f(t),\ \exp(-w\sigma t)\, \bar{H}_{p,q}^{m,n}\, [\sigma t] \rangle$$

$$= \langle f(t) \exp(-wct),\ \lambda(t) \exp[-w(\sigma - c)\, t]\, \bar{H}_{p,q}^{m,n}\, [\sigma t] \rangle. \qquad \ldots (4.9)$$

The domain space of λ is $\mathcal{D}(R)$.

We are free to choose $\lambda(t)$ such that its support is contained in $x \leq t < \infty$, where x is a fixed number satisfying $0 < x < y$. Moreover, a distribution of slow growth satisfies a boundedness condition due to Zemanian[8] (p. 109). By the boundedness property of tempered distributions there exists a constant P and a non-negative integer N such that

$$|\, F(\sigma)\, | \leq P \sup_{\substack{x \leq t < \infty \\ 0 \leq l \leq N \\ 0 \leq r \leq N}} \left| (1 + t^2)^l \left(\frac{d}{dt} \right)^r \left\{ \lambda(t) \exp[-w(\sigma - c)t]\, \bar{H}_{p,q}^{m,n}\, [\sigma t] \right\} \right|$$

$$\leq P \sup_{\substack{x \leq t < \infty \\ 0 \leq l \leq N \\ 0 \leq r \leq N}} \left| (1 + t^2)^l \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k}(t)\, \frac{d^k}{dt^k} \left\{ \exp[-w(\sigma - c)t]\, \bar{H}_{p,q}^{m,n}\, (\sigma t) \right\} \right|$$

$$\leq P \sup_{\substack{x \leq t < \infty \\ 0 \leq l \leq N \\ 0 \leq r \leq N}} \left| (1+t^2)^l \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k}(t) \sum_{j=0}^{k} \binom{k}{j} [-w(\omega - c)]^{k-j} \right.$$

$$\left. \times \exp[-w(\sigma - c)t] \left[\frac{d^j}{dt^j} \bar{H}_{p,q}^{m,n} [\sigma t] \right] \right|$$

$$\leq P \sup_{\substack{x \leq t < \infty \\ 0 \leq l \leq N \\ 0 \leq r \leq N}} \left| (1+t^2)^l \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k}(t) \sum_{j=0}^{k} \binom{k}{j} [-w(\sigma - c)]^{k-j} \right.$$

$$\left. \times \exp[-w(\sigma - c)(t-x) - w(\sigma - c)x] \frac{d^j}{dt^j} \bar{H}_{p,q}^{m,n} [\sigma t] \right| . \quad \ldots (4.10)$$

The constant P and the integer r depend only on f. Moreover, if τ is a fixed number such that $0 < \tau < x$, then

$$[w(\sigma - c)]^{k-j} \exp[-w(\sigma - c)x] \leq Q \exp[-w(\sigma - c)\tau], \quad \sigma > c$$

for each $j = 0, 1, \ldots, k$, where Q is a quantity dependent of j. Further, with $\sigma > c + 1$

$$\exp[-w(\sigma - c)(t - x)] \leq \exp[-w(t - x)], \quad t \geq x.$$

On account of the differential property of $\bar{H}_{p,q}^{m,n}[Z]$,

$$A_1 z \frac{d}{dz} \bar{H}_{p,q}^{m,n}[Z]$$

$$= \bar{H}_{p,q}^{m,n} \left[z \left| \begin{matrix} (\alpha_1 - 1, A_1), (\alpha_1, A_1; a_1 - 1), (\alpha_j, A_j; a_j)_{2,n}, \ldots, (\alpha_j, A_j)_{n+1,p} \\ (\beta_j, B_j)_{1,m}, (\beta_j, B_j; b_j)_{m+1,q} \end{matrix} \right. \right]$$

$$+ (\alpha_1 - 1) \bar{H}_{p,q}^{m,n}[Z], \quad \ldots (4.11)$$

it is observed that the behaviour of the derivative of the $\bar{H}_{p,q}^{m,n}[Z]$ as $Z \to \infty$ will depend on the order of differentiation. Again, from differentiation formula and asymptotic behaviour for $\bar{H}$-function, we see that the function

$$\frac{d^j}{dt^j} \left\{ \bar{H}_{p,q}^{m,n} \left[\sigma t \left| \begin{matrix} (\alpha_i, A_i; a_i)_{1,n}, (\alpha_i, A_i)_{n+1,p} \\ (\beta_i, B_i)_{1,m}, (\beta_i, B_i; b_i)_{m+1,q} \end{matrix} \right. \right] \right\}$$

is bounded and continuous on $0 < t < \infty$.

Thus (4.10) is dominated by

$$| F(\sigma) | \leq PQ \exp[-w(\sigma - c)\tau]$$

$$\times \sup_{\substack{x \leq t < \infty \\ 0 \leq r \leq N}} \left| (1+t^2)^l \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k}(t) \sum_{j=0}^{k} \binom{k}{j} \right.$$

$$\left. \times \exp[-w(t-x)\sigma^j \frac{d^j}{dz^j} \bar{H}_{p,q}^{m,n}[z] \right| \text{ (where } z = \sigma t). \quad \ldots (4.12)$$

But the function of t in (4.12) is bounded for all $t \geq x$, consequently

$$| F(\sigma) | \leq M \sigma^N \exp[-w(\sigma - c)\tau], \ (\sigma > 1)$$

where M is sufficiently large constant.

This completes the proof.

The initial value theorem for the distributional $\bar{H}$-function transform is as follows :

Theorem 4.2 — Let $f(t)$ be a $\bar{H}$-transformable distribution having the support in $0 \leq t < \infty$ and assume that over some neighbourhood of the origin, $f(t)$ is a regular distribution corresponding to a (Lebesgue) integrable function $h(t)$. Further assume that there exists a complex number α and a real number $\eta > -1$ such that

$$\lim_{t \to 0+} \frac{h(t)}{t^\eta} = \alpha \quad \ldots (4.13)$$

then the following result holds :

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} F(\sigma)}{\bar{H}\ (1/w)} = \alpha \quad \ldots (4.14)$$

where $F(\sigma)$ and $\bar{H}(1/w)$ are defined by (4.4) and (4.3) respectively.

PROOF : Let the neighbourhood over which $f(t)$ is a regular distribution be $-\infty < t < T$ $(T > 0)$ and let $0 < y < T$. Then $f(t)$ can be decomposed as

$$f(t) = f_1(t) + f_2(t) \quad \ldots (4.15)$$

where the supports of $f_1(t)$ and $f_2(t)$ are contained in the respective intervals $0 \leq t \leq y$ and $y \leq t < \infty$. Thus, $f_1(t)$ is a regular distribution that corresponds to $h(t)$ for $0 < t < y$. Let

$$F_1(s) \stackrel{\Delta}{=} \bar{\mathcal{H}}[f_1(t)] \text{ and } F_2(s) \stackrel{\Delta}{=} \bar{\mathcal{H}}[f_2(t)]. \quad \ldots (4.16)$$

It follows from (4.8) that

$$(w\sigma)^{\eta+1} F_2(\sigma) < M w^{\eta+1} \sigma^{r+\eta+1} \exp[-w(\sigma - c)\tau] \to 0 \text{ as } \sigma \to \infty$$

so that $$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} F_2(\sigma)}{\bar{H}\ (1/w)} = 0.$$

Since the ordinary $\bar{H}$-function transform of a (Lebesgue) integrable function is identical to the distributional $\bar{H}$-function transform of the corresponding regular distribution, we have from Theorem 4.1, that

$$\lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} F_2(\sigma)}{\bar{H}\ (1/w)} = \lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} [F_1(\sigma) + F_2(\sigma)]}{\bar{H}\ (1/w)}$$

$$= \lim_{\sigma \to \infty} \frac{(w\sigma)^{\eta+1} F_1(\sigma)}{\bar{H}\ (1/w)} = \alpha. \qquad \ldots (4.17)$$

5. A Final Value Abelian Theorem for the $\bar{H}$-Function Transform

Theorem 5.1 — Let $f(t)$ be a measurable function on $0 < t < \infty$ satisfying the condition that there exists a real number c such that $\left\{\exp(-wct)\, \bar{H}_{p,q}^{m,n}[ct]\right\}$ is absolutely convergent over $-\infty < t < \infty$. Also, if there exists a complex number α and a real number η ($\eta > -1$), such that

$$\lim_{t \to \infty} \frac{f(t)}{t^{\eta}} = \alpha \qquad \ldots (5.1)$$

then

$$\lim_{\sigma \to 0} \frac{(w\sigma)^{\eta+1} F(\sigma)}{\bar{H}\ (1/w)} = \alpha. \qquad \ldots (5.2)$$

Proof : It is clear from (5.1) that $f(t)$ is a function of slow growth. Assuming σ and y to be real positive quantities and proceeding as in the same way, we have

$$\left| (w\sigma)^{\eta+1} F(\sigma) - \alpha \bar{H}\left(\frac{1}{w}\right) \right| = (w\sigma)^{\eta+1} \left| \int_0^{\infty} \exp(-w\sigma t) \times \bar{H}_{p,q}^{m,n}[\sigma(t)]\ \{f(t) - \alpha t^{\eta}\} \right| dt$$

$$\leq (w\sigma)^{\eta+1} \int_0^{y} \left| t^{\eta} \exp(-w\sigma t)\, \bar{H}_{p,q}^{m,n}[\sigma t] \right| dt \sup_{0<t<y} \left| \frac{f(t)}{t^{\eta}} - \alpha \right|$$

$$+ w^{\eta+1} \int_{\sigma y}^{\infty} \left| X^{\eta} \exp(-wX) \cdot \bar{H}_{p,q}^{m,n}[X] \right| dX \sup_{X \leq t < \infty} \left| \frac{f(t)}{t^{\eta}} - \alpha \right|$$

(where $X = \sigma t$)

$$= I_1 + I_2. \qquad \ldots (5.3)$$

Let ε be a fixed but arbitrary positive number. In view of (5.1), y can certainly be chosen so large that I_2, which is independent of σ, can be made less than $\varepsilon/2$. After fixing y in this way, we can choose σ so small that I_1 is also less than $\varepsilon/2$. Hence (5.3) gives

$$\left| (w\sigma)^{\eta+1} F(\sigma) - \alpha \bar{H}\left(\frac{1}{w}\right) \right| < \varepsilon \text{ for small } \sigma.$$

Thus

$$\lim_{\sigma \to 0+} \frac{(w\sigma)^{\eta+1} F(\sigma)}{\bar{H}(1/w)} = \alpha \text{ as } \varepsilon \text{ is arbitrary.}$$

Hence our theorem is proved.

We can extend the preceding theorem to certain distributions as follows :

Theorem 5.2 — Let $f(t)$ be $\bar{H}$-transformable distribution in D_R' (see Zemanian[8], p. 135) and assume that $f(t)$ is a regular distribution, over some semi-infinite interval $\tau < t < \infty$, corresponding to a locally integrable function $h(t)$ satisfying the condition that

$$h(t) \exp(-wct)\, \bar{H}_{p,q}^{m,n}[ct]$$

is integrable over $-\infty < t < \infty$. Further, if there exists a complex number α and a real number $\eta > -1$ such that

$$\lim_{t \to \infty} \frac{h(t)}{t^{\eta}} = \alpha \qquad \ldots (5.4)$$

then the following result holds :

$$\lim_{\sigma \to 0+} \frac{(w\sigma)^{\eta+1} F(\sigma)}{\bar{H}(1/w)} = \alpha. \qquad \ldots (5.5)$$

PROOF : Let $f(t)$ be decomposed into

$$f(t) = f_1(t) + f_2(t)$$

where $f_2(t)$ has its support contained in the interval $y \le t < \infty$ $(y > \tau)$ and $f_1(t)$ has a bounded support that does not extend to the right of $t = y$. As, $F_1(s) \overset{\Delta}{=} \bar{\mathcal{H}}[f_1(t)]$ is entire function of s, we have

$$\lim_{\sigma \to 0+} (w\sigma)^{\eta+1} F_1(\sigma) = 0. \qquad \ldots (5.6)$$

Then, with

$$F_2(s) \overset{\Delta}{=} \bar{\mathcal{H}}[f_2(t)] \overset{\Delta}{=} \bar{\mathcal{H}}[h(t) + 1_+(t-y)]$$

where $1_+(t)$ denotes Heaviside unit step function [Zemanian[8], (p. 12)], we have

$$\lim_{\sigma \to 0+} \{(w\sigma)^{\eta+1} F(\sigma)\} = \lim_{\sigma \to 0+} \{(w\sigma)^{\eta+1} F_2(\sigma)\}$$

$$= \alpha \bar{H}(1/w)$$

with the help of (5.2). This completes the proof of Theorem 5.2.

Particular Cases

(i) Substituting $a_i = b_j = 1$ for $i = 1, ..., n$ and $j = m + 1, ..., q$ in (2.2), we obtain the results similar to that given by Saxena[7].

(ii) On the other hand, if we set $w = 0$, $a_i = b_j = 1$, for $i = 1, ..., n$ and $j = m + 1, ..., q$, the results given by Charmichael and Pathak[2] are obtained.

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