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TORSIONAL VIBRATIONS OF A FINITE PIEZOELECTRIC BONE

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Torsional vibrations of a piezoelectric solid circular cylinder of finite length belonging to crystal class 6 (bone) or 622 (β-quartz) are investigated. Curved surface and the plane ends of the cylinder are free from traction and electric potential. A series solution is found in which shear stress on plane ends is satisfied term by term and shear stress on the cylindrical surface and electric potential on both surfaces are satisfied approximately by an orthogonalization procedure. Natural frequencies of first four even and odd modes for bone are calculated. Shear stress and electric potential are presented graphically for β-quartz. It is observed that the series converges with few terms.

INTRODUCTION

Uses of Piezoelectric crystals are well known in the fields such as hydro and electric acoustics, electro-optics, communication and measurement techniques. For example quartz is used in almost all oscillators and wave filters. Particularly torsional vibration of quartz crystal is very much used in measuring shear constants of solid materials, shear viscosities and shear elasticities of liquids. Recently the applications of piezoelectric property in bone are found in curing bone fractures in bio-medical engineering. In most of the electrical devices, piezoelectric materials are used in the form of finite cylinders, plates or discs.

Torsional vibrations of an infinite circular cylinder of piezoelectric material that belongs to crystal class 622 have been discussed by Paul[1]. Torsional wave propagation in a finite piezoelectric cylindrical shell has been considered by Paul and Venkata Sarma[2]. However stresses and mode shapes are not presented numerically in the above analysis. Hutchinson[3] has studied the axisymmetric free vibration of an isotropic, elastic, finite rod. Lusher and Hardy[4] have extended the Hutchinson's method to study the free vibration of a transversely isotropic, elastic, finite rod. Following the technique of Hutchinson, torsional vibration of a finite circular cylinder of piezoelectric material that belongs to crystal class 622 has been analyzed. A series solution is obtained in which the traction free boundary conditions on the plane ends are satisfied exactly. The other boundary conditions are satisfied approximately by an orthogonalization procedure. The shear stress and potential are calculated numerically.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

A finite right circular cylinder of piezoelectric material belonging to crystal class 6 or 622 with crystallographic c axis along the axis of the cylinder is considered. The cylindrical polar coordinates, r, θ and z are referred with origin at the centre of the rod and the axis of the cylinder as z axis. The curved and end plane surfaces are given by $r = a$, $z = \pm L$ where a and L are the radius and half height of the cylinder respectively.

In case of torsional vibration, the only nonvanishing mechanical displacement $u_\theta(r, z, t)$ along the cross radial direction and the electric potential $V(r, z, t)$ are independent of θ. The piezoelectric relations (Paul and Rao[5]) can be written as

$$T_{z\theta} = c_{44}\, u_{\theta,z} + e_{14}\, V_{,r}\ ,\quad T_{r\theta} = c_{66}\,(u_{\theta,r} - u_{\theta,r}/r)$$

$$D_r = e_{14}\, u_{\theta,z} - \varepsilon_{11}\, V_{,r}\ ,\quad D_z = -\,\varepsilon_{33}\, V_{,z} \qquad \text{... (1)}$$

where $T_{z\theta}, T_{r\theta}$ are shear stresses, D_r, D_z are the electric displacements and $c_{ij}, e_{ij}, \varepsilon_{ij}$ are elastic, piezoelectric and dielectric constants respectively.

The equations of motion and Gauss equation in cylindrical polar coordinates (Paul and Raju[6]) are in the form

$$\left.\begin{aligned} &c_{66}\,(u_{\theta,rr} + u_{\theta,r}/r - u_\theta/r^2) + c_{44}\, u_{\theta,zz} + e_{14}\, V_{,rz} = \rho u_{\theta,tt} \\ &e_{14}\,(u_{\theta,rz} + u_{\theta,z}/r) - \varepsilon_{11}\,(V_{,rr} + V_{,r}/r) - \varepsilon_{33}\, V_{,zz} = 0 \end{aligned}\right] \qquad \text{... (2)}$$

where t is time and ρ the mass density. The comma followed by subscripts denotes the partial differentiation with respect to designated variable.

Since the curved surface and the end planes of the cylinder are free from traction and coated with electrodes that are shorted, the boundary conditions become

$$T_{r\theta} = 0 = V \text{ at } r = a \text{ and } T_{z\theta} = 0 = V \text{ at } z = \pm L. \qquad \text{... (3)}$$

METHODS OF SOLUTION

Two sets of solutions for eqns. (2) are obtained using the separation of variables technique.

Firstly, we seek the solutions of eqns. (2) in the form

$$u_\theta = u(r) \begin{Bmatrix} \cos(kz) \\ \sin(kz) \end{Bmatrix} \exp(-ipt) \qquad \text{... (4a)}$$

$$V = (c_{44}/e_{14})\, \varphi(r) \begin{Bmatrix} -\sin(kz) \\ \cos(kz) \end{Bmatrix} \exp(-ipt) \qquad \text{... (4b)}$$

where k, p are the wavenumber and the angular frequency respectively.

Differentiating the second equation in (2) with respect to x (a dimensionless variable $x = r/a$) and using eqn. (4) yields

$$\left.\begin{aligned}[\bar{c}_{66}\nabla^2 - \varepsilon^2 + (Ca)^2]\,u - \varepsilon\,\varphi_{,x} &= 0\\ K_{14}^2\,\varepsilon\,\nabla^2 u - (\bar{\varepsilon}_{11}\nabla^2 - \varepsilon^2)\,\varphi_{,x} &= 0\end{aligned}\right] \qquad \dots (5)$$

in which $\bar{c}_{66} = c_{66}/c_{44}$, $\bar{\varepsilon}_{11} = \varepsilon_{11}/\varepsilon_{33}$, $C^2 = \rho p^2/c_{44}$, $\varepsilon = ka$, $K_{14}^2 = e_{14}^2/(\varepsilon_{33}\,c_{44})$, $x = r/a$, and $\nabla^2 = \partial^2/\partial x^2 + (1/x)\,\partial/\partial x - (1/x)^2$, where ε is a dimensionless wavenumber.

Equation (5) can be expressed as

$$[\bar{c}_{66}\,\bar{\varepsilon}_{11}\,\nabla^2 - \varepsilon^2\,(\bar{c}_{66} + \bar{\varepsilon}_{11} + K_{14}^2 - C^2 a^2\,\bar{\varepsilon}_{11}/\varepsilon^2)\,\nabla^2 + \varepsilon^2\,(\varepsilon^2 - C^2 a^2)]$$

$$\times\ (u, \varphi_{,x}) = 0. \qquad \dots (6)$$

The solutions of eqn. (6) are

$$u = \sum_{i=1}^{2} A_i J_1(\alpha_i a x), \qquad \varphi = \sum_{i=1}^{2} A_i e_i J_0(\alpha_i a x) \qquad \dots (7)$$

where $(\alpha_i a)^2$ are the roots of the equation

$$\bar{c}_{66}\,\bar{\varepsilon}_{11}\,(\alpha a)^4 + \varepsilon^2\,(\bar{c}_{66} + \bar{\varepsilon}_{11} + K_{14}^2 - C^2 a^2\,\bar{\varepsilon}_{11}/\varepsilon^2)\ (\alpha a)^2$$

$$+ \varepsilon^2\,(\varepsilon^2 - C^2 a^2) = 0 \qquad \dots (8)$$

with $e_i = [c_{66}\,(\alpha_i a)^2 + \varepsilon^2 - (Ca)^2]/(\alpha_i a\varepsilon)$ and A_i are arbitrary constants.

Therefore u_θ, V are given by

$$u_\theta = \sum_{i=1}^{2} A_i J_1(\alpha_i a x)\begin{Bmatrix}\cos(\varepsilon y)\\ \sin(\varepsilon y)\end{Bmatrix}\exp(-ipt) \qquad \dots (9a)$$

$$V = (c_{44}/e_{14})\sum_{i=1}^{2} A_i e_i J_0(\alpha_i a x)\begin{Bmatrix}-\sin(\varepsilon y)\\ \cos(\varepsilon y)\end{Bmatrix}\exp(-ipt) \qquad \dots (9b)$$

and using eqns. (9) in (1), stresses can be calculated as

$$T_{r\theta} = -(c_{66}/a)\sum_{i=1}^{2} A_i\,(\alpha_i a)\,J_2(\alpha_i a x)\begin{Bmatrix}\cos(\varepsilon y)\\ \sin(\varepsilon y)\end{Bmatrix}\exp(-ipt) \qquad \dots (10a)$$

$$T_{z\theta} = (c_{44}/a)\sum_{i=1}^{2} A_i\,(\varepsilon - e_i\,\alpha_i a)\,J_1(\alpha_i a x)\begin{Bmatrix}-\sin(\varepsilon y)\\ \cos(\varepsilon y)\end{Bmatrix}\exp(-ipt)$$

$$\dots (10b)$$

where $y = z/a$ is a dimensionless variable.

To get the second set of solutions, u_θ and V are taken as

$$u_\theta = J_1(kr)\,\bar{u}(z)\,e^{-ipt}, \quad V = (c_{44}/e_{14})\,J_0(kr)\,\bar{\varphi}(z)\,e^{-ipt}. \qquad \dots (11)$$

Substituting eqns. (11) in eqns. (2) yield

$$\left.\begin{aligned} [a^2 D^2 - \bar{c}_{66}\,\varepsilon^2 + (Ca)^2]\,\bar{u} - \varepsilon a D\bar{\varphi} &= 0 \\ -\,K_{14}^2\,\varepsilon\, aD\bar{u} + (a^2 D^2 - \bar{\varepsilon}_{11}\,\varepsilon^2)\,\bar{\varphi} &= 0 \end{aligned}\right] \qquad \ldots (12)$$

in which $D = d/dz$.

Equations (12) can be written as

$$\{(aD)^4 - \varepsilon^2\ [\ \bar{c}_{66} + \bar{\varepsilon}_{11} + K_{14}^2 - (Ca)^2/\varepsilon^2]\ (aD)^2 - [(Ca)^2 - \bar{c}_{66}\,\varepsilon^2]\ \bar{\varepsilon}_{11}\,\varepsilon^2\}\ (\bar{u}, \bar{\varphi}) = 0. \qquad \ldots (13)$$

The solutions of eqn. (13) are

$$\bar{u} = \sum_{i=1}^{2} [B_i \cos(a\beta_i y) + C_i \sin(a\beta_i y)] \qquad \ldots (14a)$$

$$\bar{\varphi} = \sum_{i=1}^{2} d_i\,[-B_i \sin(a\beta_i y) + C_i \cos(a\beta_i y)] \qquad \ldots (14b)$$

where $(a\beta_i)^2$ are the roots of the equation

$$(a\beta)^4 + \varepsilon^2\,[\,\bar{c}_{66} + \bar{\varepsilon}_{11} + K_{14}^2 - (Ca)^2/\varepsilon^2]\ (a\beta)^2 - [(Ca)^2 - \bar{c}_{66}\,\varepsilon^2]\ \bar{\varepsilon}_{11}\,\varepsilon^2 = 0 \qquad \ldots (15)$$

with $d_i = -\,\varepsilon a\beta_i K_{14}^2/(\bar{\varepsilon}_{11}\,\varepsilon^2 + a_i^2\,\beta_i^2)$ where B_i and C_i are arbitrary constants.

Therefore u_θ and V can be written in the form like the first set of solutions, as

$$u_\theta = \sum_{i=1}^{2} B_i \left\{ \begin{matrix} \cos(a\beta_i y) \\ \sin(a\beta_i y) \end{matrix} \right\} J_1(\varepsilon x) \exp(-ipt) \qquad \ldots (16a)$$

$$V = (c_{44}/e_{14}) \sum_{i=1}^{2} d_i B_i \left\{ \begin{matrix} -\sin(a\beta_i y) \\ \cos(a\beta_i y) \end{matrix} \right\} J_0(\varepsilon x) \exp(-ipt) \qquad \ldots (16b)$$

and stresses can be found as

$$T_{r\theta} = -(c_{66}/a) \sum_{i=1}^{2} \varepsilon B_i \left\{ \begin{matrix} \cos(a\beta_i y) \\ \sin(a\beta_i y) \end{matrix} \right\} J_2(\varepsilon x) \exp(-ipt) \qquad \ldots (17a)$$

$$T_{z\theta} = (c_{44}/a) \sum_{i=1}^{2} (a\beta_i - d_i\,\varepsilon) B_i \left\{ \begin{matrix} -\sin(a\beta_i y) \\ \cos(a\beta_i y) \end{matrix} \right\} J_1(\varepsilon x) \exp(-ipt) \qquad \ldots (17b)$$

TRACTION AND POTENTIAL FREE BOUNDARY SOLUTIONS

From the boundary conditions in (3), using eqns. (9), (10), (16) and (17), it can be easily seen that the dimensionless wavenumber takes the following values :

$$[(2n - 1)/2]\pi(a/L),\ n\pi(a/L),\ \chi_{0n},\ \chi_{2n} \text{ where } n = 1, 2, 3, \ldots$$

and χ_{0n}, χ_{2n} are zeros of bessel functions of order zero and two respectively, that is, $J_0(\chi_{0n}) = 0$, $J_2(\chi_{2n}) = 0$.

Therefore the solutions can be written in the series form as follows :

$$u_\theta = \sum_n \bar{A}_n u_{\theta An} + \sum_n \bar{B}_n u_{\theta Bn} + \sum_n \bar{C}_n u_{\theta Cn} + \sum_n \bar{D}_n u_{\theta Dn} + \sum_n \bar{E}_n u_{\theta En} + \sum_n \bar{F}_n u_{\theta Fn} \qquad \ldots (18a)$$

$$V = \sum_n \bar{A}_n V_{An} + \sum_n \bar{B}_n V_{Bn} + \sum_n \bar{C}_n V_{Cn} + \sum_n \bar{D}_n V_{Dn} + \sum_n \bar{E}_n V_{En} + \sum_n \bar{F}_n V_{Fn} \qquad \ldots (18b)$$

The shear stresses can be calculated from eqns. (18a) and (18b) using eqn. (1) with same coefficients $\bar{A}_n$, $\bar{B}_n$, etc. The solutions for the mechanical displacement and electric potential are as follows :

$$u_{\theta A} = \sum_{i=1}^{2} A_i J_1(\alpha_i ax) \sin(\varepsilon_1 y) \exp(-ipt),$$

$$u_{\theta B} = \sum_{i=1}^{2} B_i J_1(\varepsilon_0 x) \sin(a\beta_i y) \exp(-ipt),$$

$$u_{\theta C} = \sum_{i=1}^{2} C_i J_1(\varepsilon_2 x) \sin(a\beta_i y) \exp(-ipt),$$

$$V_A = (c_{44}/e_{14}) \sum_{i=1}^{2} A_i e_i J_0(\alpha_i ax) \cos(\varepsilon_1 y) \exp(-ipt),$$

$$V_B = (c_{44}/e_{14}) \sum_{i=1}^{2} B_i d_i J_0(\varepsilon_0 x) \cos(a\beta_i y) \exp(-ipt),$$

$$V_C = (c_{44}/e_{14}) \sum_{i=1}^{2} C_i d_i J_0(\varepsilon_2 x) \cos(a\beta_i y) \exp(-ipt), \qquad \ldots (19)$$

and the shear stresses are given by

$$\left.\begin{aligned} T_{r\theta_A} &= -(c_{66}/a)\sum_{i=1}^{2} A_i(\alpha_i a)J_2(\alpha_i ax)\sin(\varepsilon_1 y)\exp(-ipt) \\ T_{r\theta_B} &= -(c_{66}/a)\sum_{i=1}^{2} B_i\varepsilon_0 J_2(\varepsilon_0 x)\sin(a\beta_i y)\exp(-ipt) \\ T_{r\theta_C} &= -(c_{66}/a)\sum_{i=1}^{2} C_i\varepsilon_2 J_2(\varepsilon_2 x)\sin(a\beta_i y)\exp(-ipt) \end{aligned}\right] \quad \ldots (20)$$

$$\left.\begin{aligned} T_{z\theta_A} &= (c_{44}/a)\sum_{i=1}^{2} A_i(\varepsilon_1 - e_i\alpha_i a)J_1(\alpha_i ax)\cos(\varepsilon_1 y)\exp(-ipt) \\ T_{z\theta_B} &= (c_{44}/a)\sum_{i=1}^{2} B_i(a\beta_i - d_i\varepsilon_0)J_1(\varepsilon_0 x)\cos(a\beta_i y)\exp(-ipt) \\ T_{z\theta_C} &= (c_{44}/a)\sum_{i=1}^{2} C_i(a\beta_i - d_i\varepsilon_2)J_1(\varepsilon_2 x)\cos(a\beta_i y)\exp(-ipt). \end{aligned}\right] \quad \ldots (21)$$

Similar expressions for D, E, F series can be obtained replacing sin by cos and cos by $-$ sin functions and ε_1 by ε from A series to D series.

In the above expressions $\varepsilon = n\pi(a/L)$, $\varepsilon_1 = (2n-1)(\pi/2)(a/L)$, $\varepsilon_0 = \chi_{0n}$, $\varepsilon_2 = \chi_{2n}$ where n = 1, 2, 3, ... and corresponding α_i, β_i, e_i and d_i are as in (8) and (15).

In order to satisfy the boundary conditions on $T_{z\theta}$ at $z = \pm L$ and V at $r = a$ we chosse

$$\left.\begin{aligned} A_1/A_2 &= [e_2 J_0(\alpha_2 a)]/[-e_1 J_c(\alpha_1 a)] \\ B_1/B_2 &= [(a\beta_2 - d_2\varepsilon_0)\cos(\beta_2 L)]/[-(a\beta_1 - d_1\varepsilon_0)\cos(\beta_1 L)] \\ C_1/C_2 &= [(a\beta_2 - d_2\varepsilon_2)\cos(\beta_2 L)]/[-(a\beta_1 - d_1\varepsilon_2)\cos(\beta_1 L)] \end{aligned}\right] \quad \ldots (22)$$

and similar expressions may be written for D_1/D_2, E_1/E_2 and F_1/F_2 by replacing cos by sin functions. It may be noted that the boundary condition on V at $r = a$ is satisfied for all series except C and F. The remaining boundary conditions which are to be satisfied are

$$T_{r\theta}(a, z, t) = 0 \text{ and } V(r, \pm L, t) = 0. \quad \ldots (23)$$

The boundary condition on V at $z = \pm L$ can be written in an equivalent form

$$\left.\begin{aligned} V(r, L, t) - V(r, -L, t) &= V_0(r, L, t) = 0 \\ V(r, L, t) + V(r, -L, t) &= V_e(r, L, t) = 0 \end{aligned}\right] \quad \ldots (24)$$

These boundary conditions (23) and (24) are satisfied by making $T_{r\theta}$ and V orthogonal to a complete set of functions on the boundaries :

$$\int_{-L}^{L} T_{r\theta}(a, z, t) \sin(\gamma_1 z/a)\, dz = 0 \qquad \text{... (25)}$$

$$\int_{-L}^{L} T_{r\theta}(a, z, t) \cos(\gamma z/a)\, dz = 0 \qquad \text{... (26)}$$

$$\int_{0}^{a} V_0(r, L, t) r J_0(\gamma_0 r/a)\, dr = 0 \qquad \text{... (27)}$$

$$\int_{0}^{a} V_e(r, L, t) r J_0(\gamma_0 r/a)\, dr = 0 \qquad \text{... (28)}$$

$$\int_{-L}^{L} V(a, z, t) \cos(\gamma_1 z/a)\, dz = 0 \qquad \text{... (29)}$$

$$\int_{-L}^{L} V(a, z, t) \sin(\gamma z/a)\, dz = 0 \qquad \text{... (30)}$$

where $\gamma = m\pi(a/L)$, $\gamma_1 = [(2m-1)/2]\,\pi(a/L)$, $\gamma_0 = \chi_0$, $m = 1, 2, 3, \ldots$. The necessary integrals can be found as in Watson[7].

Equations (25), (27) and (29) give

$$\hat{A}_m \bar{A}_m + \sum_n \hat{B}_{mn} \bar{B}_n = 0,$$

$$\hat{B}_m \bar{B}_m + \sum_n \hat{C}_{mn} \bar{C}_n = 0, \qquad \text{... (31)}$$

$$\sum_n C_{mn} \bar{C}_n = 0,$$

where

$$\hat{A}_m = L \sum_{i=1}^{2} A_i(\alpha_i a) J_2(\alpha_i a),$$

$$\hat{B}_{mn} = -2a\varepsilon_0 J_2(\varepsilon_0) \sin(\gamma_1 L/a) \sum_{i=1}^{2} B_i\, [(a\beta_i)/(a^2\beta_i^2 - \gamma^2)]\cos(\beta_i L),$$

$$\hat{B}_m = (a^2/2)\,J_1^2\,(\varepsilon_0) \sum_{i=1}^{2} B_i\,d_i \cos(\beta_i L),$$

$$\hat{C}_{mn} = -(a^2/2)\,J_0\,(\varepsilon_2) \sum_{i=1}^{2} C_i\,d_i \cos(\beta_i L)\,\varepsilon_2 J_1\,(\gamma_0)/(\varepsilon_2^2 - \gamma_0^2),$$

$$\bar{C}_{mn} = 2aJ_0\,(\varepsilon_2)\sin(\gamma_1 L/a) \sum_{i=1}^{2} C_i\,[(a\beta_i)/(a^2\,\beta_i^2 - \gamma^2)]\cos(\beta_i L). \qquad \text{... (32)}$$

Similar equations for D, E, F series can be obtained from eqns. (26), (28) and (30) and the coefficients from (32) by replacing sin by cos and cos by – sin functions. We can combine these equations as

$$\begin{bmatrix} \hat{A}_m & \hat{B}_{mn} & 0 \\ \vdots & \vdots & \vdots \\ 0 & \hat{B}_m & \hat{C}_{mn} \\ \vdots & \vdots & \vdots \\ 0 & 0 & \bar{C}_{mn} \end{bmatrix} \begin{bmatrix} \bar{A}_n \\ \vdots \\ \bar{B}_n \\ \vdots \\ \bar{C}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{... (33)}$$

$$\begin{bmatrix} \hat{D}_m & \hat{E}_{mn} & 0 \\ \vdots & \vdots & \vdots \\ 0 & \hat{E}_m & \hat{F}_{mn} \\ \vdots & \vdots & \vdots \\ 0 & 0 & \bar{F}_{mn} \end{bmatrix} \begin{bmatrix} \bar{D}_n \\ \vdots \\ \bar{E}_n \\ \vdots \\ \bar{F}_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{... (34)}$$

which separate into two uncoupled sets. The solutions containing A, B, C terms are called even solutions (u_θ, $T_{r\theta}$ are odd functions of z) and D, E, F terms are called odd solutions (u_θ, $T_{r\theta}$ are even functions of z) following Hutchinson[3]. The natural frequencies are found by searching the zeros of the determinants of matrices in eqns. (33) and (34). After calculating the frequencies of vibrations, u_θ, V, $T_{r\theta}$, $T_{z\theta}$ can be obtained by calculating the coefficients $\bar{A}_n$. etc., from eqns. (33) and (34) and then using eqns. (18). While calculating $\bar{A}_n$. etc., any one of them A_n may be fixed and others can be calculated, that is, u_θ, V, $T_{r\theta}$, $T_{z\theta}$ are determined as functions of r and z by the set of coefficients $\bar{A}_n$.

Numerical Analysis

The numerical analysis of the present problem is carried out for β quartz crystal and bone. The elastic, piezoelectric and dielectric constants of the β and bone are from Paul and Rao[8] and Yoon and Katz[9]. The radius of the cylinder is taken as 0.3 cm.

The convergence of the dimensionless frequencies (Ca) can be obtained by increasing the number of terms N in the series where the size of matrices in eqns. (33) and (34) is $2N \times 2N$. The first five frequencies of both odd and even modes have been calculated for β-quartz and given in Tables I and II. The ratio of half-height to radius L/a of the cylinder is taken as 1. It can be seen from the tables that five terms in the series are enough for the convergence in both odd and even modes. As pointed out by Hutchinson[3], while calculating the frequencies of vibration, all spurious roots are neglected, for which $\varepsilon_2^2 = \gamma_0^2$ and $(a\beta_i)^2 = \gamma^2$ in eqns. (32).

TABLE I

Convergence of frequencies for the first five even modes (β-quartz)

	Frequency (Ca)				
N terms	1	2	3	4	5
2	1.573122	4.713918	5.509710	6.208185	-
3	1.571754	4.715496	5.507538	6.278117	6.709993
4	1.571520	4.715460	5.506175	6.273266	6.719010
5	1.571469	4.715465	5.506333	6.273313	6.719313
6	1.571281	4.715464	5.506342	6.274063	6.719316

L/a = 1

TABLE II

Convergence of frequencies for the first five odd modes (β-quartz)

	Frequency (Ca)				
N terms	1	2	3	4	5
2	3.142685	4.232041	6.280562	6.898124	-
3	3.142587	4.240390	6.286258	6.842338	7.250380
4	3.142328	4.240322	6.287408	6.832254	7.250244
5	3.142280	4.240299	6.285879	6.833275	7.252065
6	3.142307	4.240293	6.286516	6.833215	7.252041

The first four even and odd modes are obtained for β-quartz with different L/a ratios and shown graphically in Figs. 1 and 2 respectively. For the second odd mode, the shear stress $T_{r\theta}$ and electric potential are calculated throughout the cylinder and their distributions have been shown in Figs. 3 and 4 for various z/L and r/a values. The shear stress and electric potential are normalized so that maximum shear stress is unity which is attained at $z = 0.6L$.

The error in satisfying the traction free and potential free boundary conditions is shown graphically in Fig. 5. The error in potential function is negligibly small. The maximum error on the shear stress $T_{r\theta}$ occur at the middle and plane ends of the cylinder on the curved surface. These errors can be minimized to any required precision by considering enough terms in the series.

For bone, the first four even and odd modes are computed for different L/a ratios and presented graphically in Figs. 6 and 7 respectively.

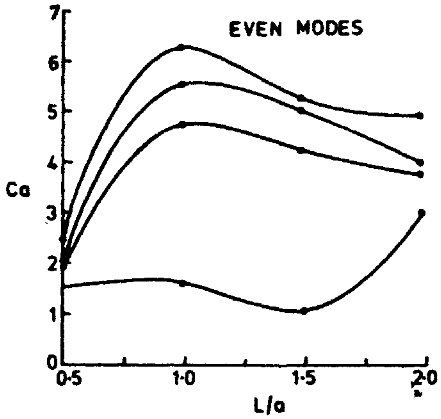

FIG. 1. Dimensionless frequency (Ca) versus half-height to radius ratio for the first four even modes of β-quartz.

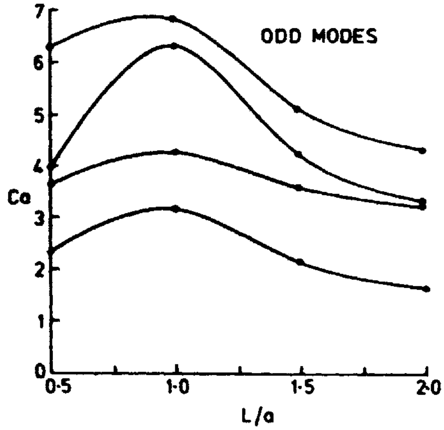

FIG. 2. Dimensionless frequency (Ca) versus half-height to radius ratio for the first four odd modes of β-quartz.

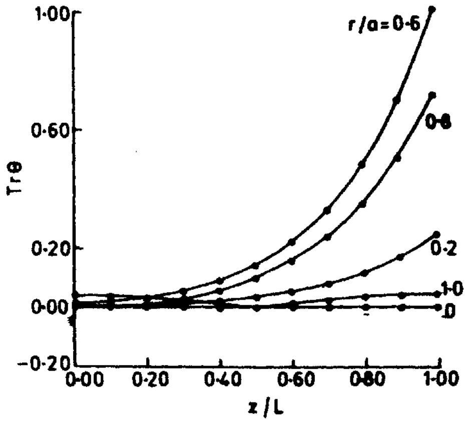

FIG. 3. Shear stress $T_{r\theta}$ of the second odd mode (β-quartz).

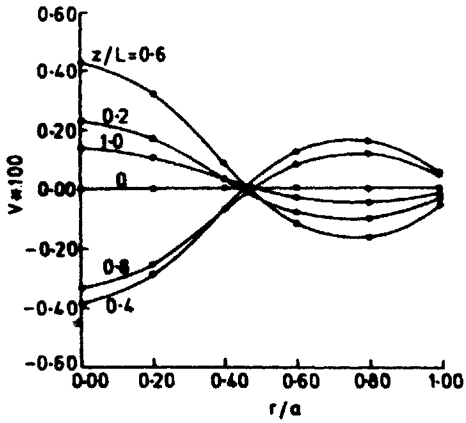

FIG. 4. Electric potential V of the second odd mode (β-quartz).

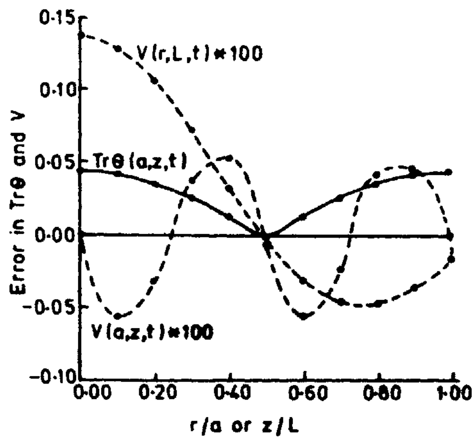

FIG. 5. Error in satisfying the shear stress and electric potential boundary conditions of the second odd mode (β-quartz).

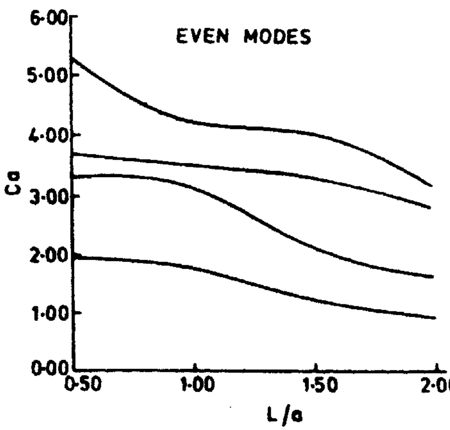

FIG. 6. Dimensionless frequency (Ca) versus half-height to radius ratio for the first four even modes of the piezoelectric bone.

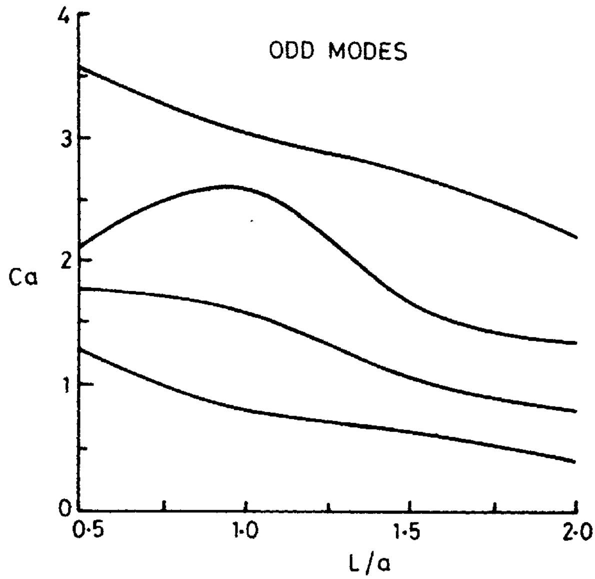

FIG. 7. Dimensionless frequency (Ca) versus half-height to radius ratio for the first four odd modes of the piezoelectric bone.

DISCUSSION

Torsional vibration of a finite piezoelectric solid cylinder that belongs to crystal class 6 or 622 is investigated. The solution is found in a series form which satisfy the traction free condition at end planes term by term. Other boundary conditions are satisfied by an orthogonalization procedure to any required precision by taking enough number of terms in the series. The numerical results are carried out for β-quartz and bone. The frequencies of vibrations and distribution of shear stress and electric potential are calculated.

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