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COMMON FIXED POINT THEOREMS FOR SOME FAMILIES OF MAPS

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We establish a lemma on an iterative procedure for four maps satisfying the Meir-Keeler type conditions. Then we give a common fixed point theorem which essentially improves and unifies the recent results of Carbone, Jungck, Kang, Kim, Pant and Rhoades. We also establish a relation between some known theorems involving (ε, δ)-type conditions and our theorem employing a contractive gauge function. Finally we extend our results to sequences of maps.

1. INTRODUCTION

Recently a number of authors have obtained common fixed point theorems involving a notion of compatibility introduced by Jungck[9] (Definition 2.1). We divide these results into two categories. The first category assumes that the maps employed satisfy some (ε, δ)-type conditions introduced by Meir and Keeler[16]. Such conditions have been recently considered by Jungck[9], Jungck *et al.*[10], Pant[17], Rao and Rao[18, 19] and Rhoades *et al.*[22]. The second category assumes that the maps satisfy some inequalities involving a contractive gauge function Φ from $\mathbb{R}_+$, the nonnegative reals, into $\mathbb{R}_+$. In general, Φ is required to be nondecreasing, upper semicontinuous (or, equivalently, nondecreasing and continuous from the right) and such that $\Phi(t) < t$ for all $t > 0$. Such a class of functions was introduced by Browder[2]. This category includes the recent results of Carbone *et al.*[3], Kang and Kim[11], Kang and Rhoades[12] and, for weakly commuting maps, Sessa *et al.*[23].

In section 3 we establish a theorem which belongs to the second of the above categories. However it is general enough to yield all of the theorems mentioned above; in particular those from the first category. Such a unified approach is possible thanks to a simple lemma on some subsets of the plane which is established in section 4.

Finally, we demonstrate how the results of section 3 can be extended to sequences of maps in order to generalize, among others, theorems of Chaterji[5] and Sessa *et al.*[23].

2. A Basic Lemma

The lemma of this section is fundamental for our further considerations. It gives us a good initial position from which to obtain several fixed point theorems for families of maps. We start by recalling the following definition given by Jungck[9] (Definition 3.2).

Definition 2.1 — Let A, B, S and T be selfmaps of a set X such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. For $x_0 \in X$, any sequence $\{y_n\}$ defined by $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for $n \in \mathbb{N}$, the positive integers, is called an S, T-iteration of x_0 under A and B.

Lemma 2.2 — Let A, B, S and T be selfmaps of a metric space (X, d) such that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. Assume further that

given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all x, y in

$$X,\ \varepsilon < M(x, y) < \varepsilon + \delta \text{ implies that } d(Ax, By) \leq \varepsilon; \quad \text{... (1)}$$

$$\text{for all } x, y \text{ in } X \text{ with } M(x, y) > 0,\ d(Ax, By) < M(x, y) \quad \text{... (2)}$$

where

$$M(x, y) := \max \{d(Sx, Ty),\ d(Ax, Sx),\ d(By, Ty),\ [d(Ax, Ty) + d(By, Sx)]/2\}. \quad \text{... (3)}$$

Then for each x_0 in X any sequence $\{y_n\}$, being an S, T-iteration of x_0 under A and B, is a Cauchy sequence.

Proof : Fix an $x_0 \in X$. For $n \in \mathbb{N}$ define $d_n := d(y_n, y_{n+1})$. We shall divide the proof into three parts.

I. We shall show that if for some k in $\mathbb{N}$ $d_{k+1} > 0$, then

$$d_{k+1} < d_k. \quad \text{... (4)}$$

(a) Assume that $d_{2k} > 0$ for some k in $\mathbb{N}$. Then $M(x_{2k}, x_{2k-1}) > 0$; for otherwise, we get by (3) that $Ax_{2k} = Bx_{2k-1}$, i.e. $y_{2k+1} = y_{2k}$ so $d_{2k} = 0$, a contradiction. Hence and by (2) we have that

$$d_{2k} = d(Ax_{2k}, Bx_{2k-1}) < M(x_{2k}, x_{2k-1}). \quad \text{... (5)}$$

Next,
$$\begin{aligned} M(x_{2k}, x_{2k-1}) &= \max \{d(Sx_{2k}, Tx_{2k-1}),\ d(Ax_{2k}, Sx_{2k}), \\ &\qquad d(Bx_{2k-1}, Tx_{2k-1}),\ [d(Ax_{2k}, Tx_{2k-1}) + d(Bx_{2k-1}, Sx_{2k})]/2\} \\ &\leq \max \{d(y_{2k}, y_{2k-1}),\ d(y_{2k+1}, y_{2k}), d(y_{2k+1}, y_{2k-1})/2\} \\ &\leq \max \{d_{2k-1}, d_{2k}, (d_{2k} + d_{2k-1})/2\} = \max \{d_{2k-1}, d_{2k}\}. \end{aligned}$$

Hence and by (5) we get that $d_{2k} < \max \{d_{2k-1}, d_{2k}\}$ which implies that $d_{2k} < d_{2k-1}$.

(b) Assume that $d_{2k+1} > 0$ for some k in $\mathbb{N}$. Using a similar argument as in (a) one can verify that $d_{2k+1} < d_{2k}$.

(c) Combining the results of (a) and (b) we may conclude that (4) holds. Hence, if for some k in $\mathbb{N}$ $d_k = 0$ then $d_{k+1} = 0$ which leads to the conclusion that $d_n = 0$ for all $n \geq k$ so $y_n = y_k$ for such n. Obviously, in such a case $\{y_n\}$ is a Cauchy sequence.

II. We shall show that $\lim_n d_n = 0$. By I(c) this is the case if $d_k = 0$ for some k in $\mathbb{N}$. So assume that $d_n > 0$ for all n in $\mathbb{N}$. Then it follows from (4) that $\{d_n\}$ is strictly decreasing, hence convergent to some d in $\mathbb{R}_+$. Suppose $d > 0$. By (1), there exists a suitable $\delta(d)$: Then, for some k in $\mathbb{N}$ and all $n \geq k$, $d < d_n < d + \delta(d)$. In particular, $d < M(x_{2k}, x_{2k-1}) < d + \delta(d)$ since $M(x_{2k}, x_{2k-1}) = \max\{d_{2k}, d_{2k-1}\}$. By (1) we get that $d_{2k} = d(Ax_{2k}, Bx_{2k-1}) \leq d$, a contradiction. This proves that $d = 0$.

III. We shall show that $\{y_n\}$ is a Cauchy sequence. Fix an $\varepsilon > 0$. By (1), there exists a suitable δ. It can be assumed that $\delta < \varepsilon$. Since $\lim_n d_n = 0$ there exists k in $\mathbb{N}$ such that

$$d_n < \frac{1}{4}\delta \text{ for all } n \geq k. \qquad \dots (6)$$

Without loss a generality we may assume that k is even. We shall apply induction to show that

$$d(y_k, y_{k+i}) < \varepsilon + \frac{1}{2}\delta \text{ for all } i \text{ in } \mathbb{N}. \qquad \dots (7)$$

By (6), (7) holds for $i = 1$. Assuming (7) to hold for $i = 1, 2, \dots, n$ where $n \geq 1$, we shall prove it for $i = n + 1$. Let us consider two cases.

(a) n is odd. Then, by the triangle inequality, we have

$$d(y_k, y_{k+n+1}) \leq d_k + d(y_{k+1}, y_{k+n+1}). \qquad \dots (8)$$

Since $k + 1$ is odd and $k + n + 1$ is even we have that $d(y_{k+1}, y_{k+n+1}) = d(Ax_k, Bx_{k+n})$. Denote $x = x_k$ and $y = x_{k+n}$. Then, by (6) and the induction hypothesis we get that

$$d(Sx, Ty) = d(y_k, y_{k+n}) < \varepsilon + \frac{1}{2}\delta, \quad d(Ax, Sx) = d(y_{k+1}, y_k) < \frac{1}{4}\delta,$$

$$d(By, Ty) = d(y_{k+n+1}, y_{k+n}) < \frac{1}{4}\delta, \quad \frac{1}{2}[d(Ax, Ty) + d(Sx, By)]$$

$$\leq d(Sx, Ty) + \frac{1}{2}[d(Ax, Sx) + d(By, Ty)] < \varepsilon + \frac{1}{2}\delta + \frac{1}{4}\delta < \varepsilon + \delta.$$

Hence, $M(x, y) < \varepsilon + \delta$. If simultaneously $M(x, y) > \varepsilon$ then, by (1), we get that $d(Ax, By) \leq \varepsilon$; if $0 < M(x, y) \leq \varepsilon$ then, by (2), we have that $d(Ax, By) \leq \varepsilon$; if $M(x, y) = 0$ then one can verify that $d(Ax, By) = 0$. Thus in all cases we have that

$d(Ax, By) \leq \varepsilon$, i.e. $d(y_{k+1}, y_{k+n+1}) \leq \varepsilon$. Hence and by (6) and (8), we get that $d(y_k, y_{k+n+1}) < \varepsilon + \frac{1}{4}\delta$ so (7) holds for $n + 1$.

(b) n is even. Then, by the triangle inequality,

$$d(y_k, y_{k+n+1}) \leq d(y_k, y_{k+1}) + d(y_{k+1}, y_{k+n}) + d(y_{k+n}, y_{k+n+1}). \qquad \dots (9)$$

Since $k + 1$ is odd and $k + n$ is even we have that $d(y_{k+1}, y_{k+n}) = d(Ax_k, Bx_{k+n-1})$. Similarly as in the part (a) using (6) and (7) with $i = n - 1$ one can verify that $M(x_k, x_{k+n-1}) < \varepsilon + \delta$. In a consequence, $d(y_{k+1}, y_{k+n}) \leq \varepsilon$ so by (6) and (9) we get that $d(y_k, y_{k+n+1}) < \varepsilon + \frac{1}{2}\delta$.

Thus in both the cases (a) and (b) we get that (7) holds for $i = n + 1$. By the induction principle, the proof of (7) is completed. Hence we get that $d(y_k, y_{k+i}) < \frac{3}{2}\varepsilon$ for all i in $\mathbb{N}$ since $\delta < \varepsilon$. Obviously, this implies that $\{y_n\}$ is a Cauchy sequence.

Remark 1 : The (ε, δ)-type conditions were introduced by Meir and Keeler[16] who established a fixed point theorem under the following hypothesis on a map f involved :

for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all x, y in X,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies that } d(fx, fy) < \varepsilon.$$

Many authors generalized the above result replacing $d(x, y)$ and $d(fx, fy)$ by some other terms (for details, see a paper of Rhoades *et al.*[22] and references therein). Keeping such an idea, it suggests to consider the following extension of the Meir-Keeler condition on four maps :

for any $\varepsilon > 0$ there exists a $\delta > 0$ such that, for all x, y in X,

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \text{ implies that } d(Ax, By) < \varepsilon. \qquad \dots (10)$$

Our conditions (1) and (2) are however slightly more general than (10). To ensure about it, consider the following example :

Example — Let $X := \mathbb{R}_+$ and, for $x, y \in X$, $d(x, y) := \max\{x, y\}$ if $x \neq y$ and $d(x, y) := 0$ if $x = y$. One can verify that (X, d) is a complete metric space. Further, define maps A, B, S and T as follows : $S := T := I$, the identity map on X; $A0 := 0$, $Ax := \frac{1}{n+1}$ for $\frac{1}{n+1} < x \leq \frac{1}{n}$ $(n \in \mathbb{N})$ and $Ax := 1$ for $x > 1$; $B := A$. Then one can verify that $M(x, y) = d(x, y)$ and that (2) holds. We shall show that (1) holds. Fix an $\varepsilon > 0$. If $\varepsilon \geq 1$ then δ can be taken arbitrarily since $d(Ax, By) \leq 1$ for all x, y in X. So assume that $0 < \varepsilon < 1$. Then, for some n in $\mathbb{N}$, $\frac{1}{n+1} \leq \varepsilon < \frac{1}{n}$. Let $\delta := \frac{1}{n} - \varepsilon$. If $\varepsilon < M(x, y) < \varepsilon + \delta$ then we have that $\frac{1}{n+1} < \max\{x, y\}$

$< \frac{1}{n}$. Hence we may conclude that $d(Ax, By) \leq \frac{1}{n+1} \leq \varepsilon$. Thus (1) holds. Now take $\varepsilon := 1$. Then, for any $\delta > 0$, points $x := 0$ and $y := 1 + \frac{\delta}{2}$ satisfy the inequality $\varepsilon \leq M(x, y) < \varepsilon + \delta$ but $d(Ax, By) = 1 = \varepsilon$ so (10) is not satisfied.

Remark 2 : In case in which $A = B$ and $S = T = I$, conditions (1) and (2) are equivalent to those introduced by Matkowski[15]. We refer the reader to a paper of Jachymski[8] in which there is a detailed analysis of this case. Condition (1) with $A = B$ and $M(x, y)$ replaced by $d(x, y)$ was examined by Rao and Rao[18].

Remark 3 : In view of the following example, Lemma 2.2 would not be valid if one changed the definition of $M(x, y)$ by taking in it the term $\max\{d(Ax, Ty), d(By, Sx)\}$ instead of $[d(Ax, Ty) + d(By, Sx)]/2$:

Example — Let $X := \mathbb{N}$, endowed with the euclidean metric and define $S := T := I$, $Ax := Bx := x + 1$ $(x \in \mathbb{N})$. Then, for $x, y \in \mathbb{N}$, $d(Ax, By) = |x-y|$ while $\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty), d(By, Sx)\} = |x-y| + 1$. Given $\varepsilon > 0$ it suffices to take $\delta := 1$ to have that (1) and (2) hold for such a changed $M(x, y)$. However, for each x_0 in X, the S, T-iteration of x_0 under A and B is not a Cauchy sequence.

Remark 4 : One can verify that Lemma 2.2 generalizes a result of Jungck[9] (Lemma 3.1).

Remark 5 : It follows from the proof of Lemma 2.2 (see part III) that (1) and (2) imply the following condition :

given $\varepsilon > 0$, there exists a $\delta > 0$ such that, for all x, y in X

$$0 \leq M(x, y) < \varepsilon + \delta \text{ implies that } d(Ax, By) \leq \varepsilon. \qquad \text{... (11)}$$

In fact, it can be checked that the conjunction (1) $\wedge$ (2) is equivalent to (11) $\wedge$ (2).

3. Fixed Point Theorems Involving Contractive Gauge Functions

In this section we consider families of Φ-contraction maps where Φ is a function from $\mathbb{R}_+$ into $\mathbb{R}_+$. We start with the following lemma on some real functions.

Lemma 3.1 — Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$. Then we have :

(i) If Φ is upper semicontinuous from the right and $\Phi(t) < t$ for all $t > 0$ then it fulfills the following condition :

given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each $t > 0$,

$$\varepsilon \leq t < \varepsilon + \delta \text{ implies that } \Phi(t) < \varepsilon;$$

(ii) if Φ is nondecreasing and $\lim_n \Phi^n(t) = 0$ for each t in $\mathbb{R}_+$ (Φ^n denotes the nth iterate of Φ) then it fulfills the following condition :

given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for each $t > 0$,

$$\varepsilon < t < \varepsilon + \delta \text{ implies that } \Phi(t) \leq \varepsilon.$$

Moreover, $\Phi(t) < t$ for all $t > 0$.

PROOF : Part (i) was observed by Meir and Keeler[16]. To prove (ii) suppose, on the contrary, that there exists an $\varepsilon > 0$ and a sequence $\{t_n\}$ with $t_n \to \varepsilon^+$ and $\Phi(t_n) > \varepsilon$. By the monotonicity of Φ, we get that $\Phi(t) > \varepsilon$ for all $t > \varepsilon$. Hence, by an easy induction, we obtain that $\Phi^n(t) > \varepsilon$ for all $t > \varepsilon$, which contradicts the assumption $\lim_n \Phi^n(t) = 0$ for all t in $\mathbb{R}_+$. The proof of (ii) is now complete since the fact that $\Phi(t) < t$ for all $t > 0$ was proved by Matkowski[14].

Remark : Classes of functions occurring in (i) and (ii) of Lemma 3.1 were introduced, respectively, by Boyd and Wong[1], and Matkowski[14].

Corollary 3.2 — Let A, B, S and T be selfmaps of a metric space (X, d) satisfying :

$$A(X) \subseteq T(X) \text{ and } B(X) \subseteq S(X), \qquad \ldots (12)$$

$$d(Ax, By) \leq \Phi[M(x, y)] \text{ for all } x, y \text{ in } X, \qquad \ldots (13)$$

where M is the same as in Lemma 2.2 and Φ is a function which is either upper semicontinuous from the right and $\Phi(t) < t$ $(t > 0)$, or nondecreasing and $\lim_n \phi^\nu(t) = 0$ $(t \in \mathbb{R}_+)$.

Then, for each x_0 in X, an S, T-iteration of x_0 under A and B is a Cauchy sequence.

PROOF : Apply Lemmas 2.2 and 3.1.

Let us recall that two selfmaps A and S of a metric space (X, d) are said to be compatible if $\lim_n d(ASx_n, SAx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_n Ax_n = \lim_n Sx_n = y$ for some y in X (Jungck[9], Def. 2.1).

Theorem 3.3 — Let A, B, S and T be selfmaps of a complete metric space (X, d) such that A, S and B, T are compatible, one of A, B, S or T is continuous and (12) and (13) hold with a function Φ being upper semicontinuous (not necessarily monotonic) and such that $\Phi(t) < t$ for all $t > 0$.

Then A, B, S and T have a unique common fixed point z in X and, for any x_0 in X, each S, T-iteration of x_0 under A and B converges to the point z.

PROOF : Fix an $x_0 \in X$. Let $\{y_n\}$ be an S, T-iteration of x_0 under A and B. By Corollary 3.2 and the completeness of (X, d), $\{y_n\}$ converges to some z in X. We shall show that z is a desirable point. Uniqueness of the fixed point follows easily from (13). In a sequel, we consider three cases.

Case I — Assume that S is continuous.

(a) We show that $z = Sz$. By the definition of $\{y_n\}$, we have that $y_{2n} = Sx_{2n}$ and $y_{2n+1} = Ax_{2n}$, $(n \in \mathbb{N})$ so since $\lim_n y_n = z$ we get that $\lim_n Ax_{2n} = \lim_n Sx_{2n} = z$. Then, by the continuity of S, we have that $\lim_n SAx_{2n} = Sz$. Since A and S are compatible we may conclude that $\lim_n ASx_{2n} = Sz$. By putting in (13) $x = Sx_{2n}$ and $y = x_{2n-1}$ we get that

$$d(ASx_{2n}, Bx_{2n-1}) \le \Phi\,[\max\{d(SSx_{2n}, Tx_{2n-1}), d(ASx_{2n}, SSx_{2n}),\ d(Bx_{2n-1}, Tx_{2n-1}),\ [d(ASx_{2n}, Tx_{2n-1}) + d(Bx_{2n-1}, SSx_{2n})]/2\}]. \quad \text{... (14)}$$

By the definition of $\{y_n\}$, we have that $Bx_{2n-1} = y_{2n}$ and $Tx_{2n-1} = y_{2n-1}$. Since $\lim_n ASx_{2n} = \lim_n S^2 x_{2n} = Sz$, we may conclude that $\lim_n M(SX_{2n}, x_{2n-1}) = d(z, Sz)$. Simultaneously, the left side of (14) converges to $d(z, Sz)$. Denote $r := d(z, Sz)$. By (14), we may conclude that $r \le \lim_{t\to r} \sup\ \Phi(t) \le \Phi(r)$ since Φ is upper semiconti- nous. Hence $r = 0$; for otherwise we get $r \le \Phi(r) < r$, a contradiction. Thus z is a fixed point of S.

(b) We show that $z = Az = Bz = Tz$. By putting in (13) $x = z$ and $y = x_{2n-1}$ we get that

$$d(Az, Bx_{2n-1}) \le \Phi\,[\max\{d(z, Tx_{2n-1}),\ d(Az, z),\ d(Bx_{2n-1}, Tx_{2n-1}),\ [d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, z)]/2\}], \quad \text{... (15)}$$

since $z = Sz$. Suppose that $r := d(Az, z) > 0$. Since

$$\lim_n d(z, Tx_{2n-1}) = \lim_n d(Bx_{2n-1}, Tx_{2n-1}) = 0$$

and $$\lim_n [d(Az, Tx_{2n-1}) + d(Bx_{2n-1}, z)]/2 = r/2,$$

we may conclude that, for sufficiently large n, $M(z, x_{2n-1}) = r$ and then, by (15), $d(Az, Bx_{2n-1}) \le \Phi(r)$. Hence letting $n \to \infty$ we get that $r \le \Phi(r) < r$, a contradiction. Thus $r = 0$ so z is a fixed point of A.

Since $A(X) \subseteq T(X)$, there exists a point w in X such that $z = Az = Tw$. Then, by (13), we get that

$$d(z, Bw) = d(Az, Bw) \le \Phi\,[M(z, w)] = \Phi[d(z, Bw)].$$

Hence $z = Bw$. Thus $Bw = Tw$, which implies that $TBw = BTw$ because, by Proposition 2.2 of Jungck[9], B and T being compatible commute at their coincidence points. Hence $Tz = Bz$. By putting in (13) $x = y = z$ we get that $d(Az, Bz) \le \Phi\,[M(z, z)]$. Since $z = Sz = Az$ and $Tz = Bz$, we get that $M(z, z) = d(z, Bz) = d(Az, Bz)$. Then the above inequality implies that $d(z, Bz) = 0$, i.e. z is a fixed point of B. Since $Tz = Bz$, z is also a fixed point of T. So finally z is a common fixed point of A, B, S and T.

Case II — Assume that A is continuous.

We give only a sketch of a proof that then z is a common fixed point of A, B, S and T, since this proof is quite similar to the preceding one. In particular, similarly as in part I(a) one can obtain that $\lim_n SAx_{2n} = \lim_n A^2 x_{2n} = Az$. By putting in (13) $x = Ax_{2n}$ and $y = x_{2n-1}$ and using the same argument as in I(a) we obtain that $z = Az$. Since $A(X) \subseteq T(X)$, there exists a point w such that $z = Az = Tw$. By putting in (13) $x = Ax_{2n}$ and $y = w$ one can deduce that $z = Bw$ so $Bw = Tw$ and thus, by the compatibility of B and T, $BTw = TBw$, i.e. $Bz = Tz$. By putting in (13)

$x = x_{2n}$ and $y = z$ one can obtain that $z = Bz$ so also $z = Tz$. Since $B(X) \subseteq S(X)$, there exists a point v such that $z = Bz = Sv$. By putting in (13) $x = v$ and $y = z$ we get that $Av = z$. Hence $Av = Sv$, and by the compatibility of A and S, we obtain that $Az = Sz$. Since $z = Az$, z is a fixed point of S. So finally z is a common fixed point of A, B, S and T.

Case III — Assume that one of the maps T or B is continuous. We have proven Theorem 3.3 in the cases when the first or third of considered maps is continuous. To finish the proof observe that, if a quaternion of maps (A, B, S, T) satisfies the assumptions of Theorem 3.3, then so also does (B, A, T, S). To see this, interchange x and y in (13) and use the symmetry of d to obtain the following inequality :

$$d(Bx, Ay) \leq \Phi\,[\max\{d(Tx, Sy), d(Bx, Tx), d(Ay, Sy), [d(Bx, Sy) + d(Ay, Tx)]/2\}],$$

for all x, y in X. Thus the case when T or B is continuous reduces to I or II, respectively. This completes the proof.

Corollary 3.4 (Kang and Kim[11], Theorem 3.1) — Let the assumptions of Theorem 3.3 be fulfilled with Φ being linear, i.e. $\Phi(t) = ht$ for some h in $[0, 1)$ and all t in $\mathbb{R}_+$. Then A, B, S and T have a unique common fixed point.

Corollary 3.5 (Kang and Rhoades[12], Theorem 2.3) — Let S and T be surjective maps of (X, d) (the inclusions $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ are then automatically satisfied) and let the assumptions of Theorem 3.3 hold with $M(x, y)$ replaced by $d(Sx, Ty)$. Assume additionally that the function Φ is nondecreasing. Then A, B, S and T have a unique common fixed point.

The following lemma enables us to show that Theorem 3.3 includes also the result of Carbone *et al.*[3].

Lemma 3.6 (Rhoades and Watson[20], Lemma 1) — Assume that a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, $\Phi(t) < t$ for all $t > 0$ and the function $t \to t/(t - \Phi(t))$ is nonincreasing on $(0, \infty)$. Then Φ is continuous.

Corollary 3.7 (Carbone *et al.*[3], Theorem 1) — Let the assumptions of Theorem 3.3 be fulfilled with $A = B$, $S = T$ and a function Φ as in Lemma 3.6. Further, let $\int_0^z t/(t - \Phi(t))\, dt < \infty$ for each $z > 0$. Then A and S have a unique common fixed point.

Remark 1 : In fact, Carbone *et al.*[3] assumed the stronger condition that the maps A and S are weakly commuting, i.e. $d(ASx, SAx) \leq d(Ax, Sx)$ for all x in X. However as was observed by the referee of their paper, this result could be extended to compatible maps.

Remark 2 : The assumption of continuity of one of the involved maps cannot be dropped in Theorem 3.3. A suitable example has been given by Kang and Kim[11] (Example 3.2).

We conclude this section by examining the case when $S = T = I$, the identity map on X. Then the inclusions $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ as well as compatibility conditions for A, S and B, T are automatically satisfied. The following theorem shows that, in such a case, it is possible to extend the class of contractive gauge functions involved.

Theorem 3.8 — Let A and B be selfmaps of a complete metric space (X, d) such that, for all x, y in X,

$$d(Ax, By) \leq \Phi\,[\max\,\{d(x,y), d(x,Ax), d(y,By), [d(Ax,y)+d(By,x)]/2\}], \qquad \ldots (16)$$

where a function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies one of the following conditions :

(i) Φ is upper semicontinuous from the right and $\Phi(t) < t$ for all $t > 0$

or

(ii) Φ is nondecreasing and $\lim_n \Phi^n(t) = 0$ for all t in $\mathbb{R}_+$.

Then A and B have a unique common fixed point z and, for each x_0 in X, the sequence $\{x_n\}$ defined by $x_{2n-1} := Ax_{2n-2}$ and $x_{2n} := Bx_{2n-1}$ $(n \in \mathbb{N})$ converges to z.

PROOF : The convergence of $\{x_n\}$ follows from Corollary 3.2 and the completeness of (X, d). Let $z := \lim_n x_n$. Using the same argument as in part I(b) of a proof of Theorem 3.3, we obtain that z is a unique common fixed point of A and B. (Observe that in I(b) we employ only the inequality $\Phi(t) < t$ for all $t > 0$; the other properties of Φ are unnecessary in this part of the proof.)

Remark 1 : By putting in Theorem 3.8 $A = B$ and assuming that the function Φ satisfies (i), we get Theorem 2 of Leader[13].

Remark 2 : Put $A = B$ in Theorem 3.8 and assume that Φ satisfies (ii). Then we get an improvement of Theorem 6 of Jachymski[7], proved under an additional hypothesis that $\lim_{t \to \infty} \inf\,(t - \Phi(t)) > 0$. Simultaneously, replacing the right side of (16) by $d(x, y)$, we obtain Theorem 1.2 of Matkowski[14].

4. Fixed Point Theorems Involving (ε, δ)-Type Conditions

This section is devoted to studying a usefulness of condition (10) for obtaining fixed point theorems. As was observed by Rao and Rao[19] (Example 1.7), this condition does not ensure the existence of a fixed point even if (X, d) is compact and $S = T = I$ and $A = B$. This contradicts Theorem 1 of Rhoades *et al.*[22] and Theorem 1 of Rhoades[21]. Thus some additional hypothesis must be used to obtain a fixed point theorem. We shall discuss here two ways indicated by several authors. One is to add some assumptions on a function δ which occurs in (10). For example, Jungck[9] and Jungck *et al.*[10] require that δ be lower semicontinuous. Then, however, as was shown by Jachymski[8] (Theorem 1), there exists an upper semicontinuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ is nondecreasing, $\Phi(t) < t$ for all $t > 0$ and condition (13) holds (In fact, these conditions are then equivalent). Thus the theorem of Jungck *et al.*[10] can be deduced from our Theorem 3.3.

On the other hand, Pant[17] assumes that δ is nondecreasing. He uses condition (10) with $M(x, y)$ replaced by $m(x, y) = \max\,\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}$.

We shall show that our Theorem 3.3. yields also the result of Pant[17]. The idea of the proof is taken from the paper of Hegedüs and Szilágyi[6].

Lemma 4.1 — Assume that Q is a subset of $\mathbb{R}_+^2$ with the following property : there exists a nondecreasing function $\delta : (0, \infty) \to (0, \infty)$ such that,

for any $\varepsilon > 0$, $(x, y) \in Q$ and $0 \le x < \varepsilon + \delta(\varepsilon)$ imply that $y < \varepsilon$. ... (17)

Then

there exists an upper semicontinuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ is nondecreasing, $\Phi(t) < t$ for all $t > 0$ and $(x, y) \in Q$ implies that $y \le \Phi(x)$. ... (18)

PROOF : By Lemma 1 of Hegedüs and Szilágyi[6], (18) is equivalent to the following condition :

for any $\varepsilon > 0$, there exists a $\beta(\varepsilon) > 0$ and an $\eta(\varepsilon)$, $0 < \eta(\varepsilon) < \varepsilon$ such that, $0 \le x < \varepsilon + \beta(\varepsilon)$ and $(x, y) \in Q$ imply that $y \le \eta(\varepsilon)$. ... (19)

We shall verify that (19) holds. Fix an $\varepsilon > 0$. By the monotonicity of δ, we have that

$$\sup \{t + \delta(t) : t \in (0, \varepsilon) = \lim_{t \to \varepsilon^-} (t + \delta(t)) > \varepsilon.$$

Hence there exists an $\eta(\varepsilon) \in (0, \varepsilon)$ such that $\eta(\varepsilon) + \delta[\eta(\varepsilon)] > \varepsilon$. Define $\beta(\varepsilon) : = \eta(\varepsilon) + \delta[\eta(\varepsilon)] - \varepsilon$. Then $\beta(\varepsilon) > 0$. Now if $0 \le x < \varepsilon + \beta(\varepsilon)$ and $(x, y) \in Q$ then $0 \le x < \eta(\varepsilon) + \delta[\eta(\varepsilon)]$ so, by (17), $y < \eta(\varepsilon)$. Thus (19) holds.

Proposition 4.2 — Let A, B, S and T be selfmaps of a metric space (X, d). For $x, y \in X$ define

$$m(x, y) : = \max \{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}.$$

Assume there exists a nondecreasing function $\delta : (0, \infty) \to (0, \infty)$ such that

for any $\varepsilon > 0$ and x, y in X, $\varepsilon \le m(x, y) < \varepsilon + \delta(\varepsilon)$ implies that $d(Ax, By) < \varepsilon$. ... (20)

Then there exists an upper semicontinuous function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that Φ is nondecreasing, $\Phi(t) < t$ for all $t > 0$ and $d(Ax, By) \le \Phi[m(x, y)]$ for all x, y in X.

PROOF : Condition (20) implies that $d(Ax, By) < m(x, y)$ for all x, y in X with $m(x, y) > 0$. Moreover, it follows from the definition of $m(x, y)$ that if $m(x, y) = 0$ then $d(Ax, By) = 0$. Thus for any $\varepsilon > 0$, the inequality $d(Ax, By) < \varepsilon$ holds for all x, y in X with $0 \le m(x, y) < \varepsilon + \delta$. To finish the proof it suffices to apply Lemma 4.1 putting $Q := \{(m(x, y), d(Ax, By)) : x, y \in X\}$.

Remark : It follows from Proposition 4.2 that the theorem of Pant[17] is a special case of our Theorem 3.3.

Another way to obtain a fixed point theorem was chosen by Rao and Rao[19] (Theorem 1.2). They employed the Meir-Keeler type condition with $M(x, y)$ replaced by the term max $\{d(Sx, Sy), [d(Ax, Sx) + d(By, Sy)]/2, [d(Ax, Sy) + d(By, Sx)]/2\}$. In this case the function δ need be neither continuous nor monotonic. However an additional inequality is required. In the case when S is the identity map this inequality is automatically satisfied. So, in the sequel, for the sake of simplicity, we shall restrict our attention to the case when $S = I$. The following theorem unifies and extends Theorem B of Rao and Rao[18] and Theorem 1.2 of the same authors[19] (here for the case $S = I$).

Theorem 4.3 — Let A and B be selfmaps of a complete metric space (X, d). For $x, y \in X$, define

$$m(x, y) = \max \{d(x, y), [d(x, Ax) + d(By, y)]/2, [d(Ax, y) + d(By, x)]/2\}.$$

Assume further that

$$\text{for any } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that, for any } x, y \text{ in } X,\ \varepsilon < m(x, y) < \varepsilon + \delta \text{ implies that } d(Ax, By) \leq \varepsilon \quad \ldots (21)$$

and

$$d(Ax, By) < m(x, y) \text{ for all } x, y \text{ in } X \text{ with } m(x, y) > 0. \quad \ldots (22)$$

Then A and B have a unique common fixed point z and, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by $x_{2n-1} := Ax_{2n-2}$ and $x_{2n} := Bx_{2n-1}$ $(n \in \mathbb{N})$ converges to z.

PROOF : Using a similar argument as in the proof of Proposition 4.2 one can verify that (21) and (22) imply that,

$$\text{for any } \varepsilon > 0, \text{ there exists a } \delta > 0 \text{ such that, for all } x, y \text{ in } X,\ 0 \leq m(x, y) < \varepsilon + \delta \text{ implies that } d(Ax, By) \leq \varepsilon. \quad \ldots (23)$$

Now it can be easily checked that (22) and (23) imply that (1) and (2) hold with $S = T = I$ so, by Lemma 2.2, for any x_0 in X the above defined sequence $\{x_n\}$ is convergent. Fix an $x_0 \in X$ and define $z := \lim_n x_n$. Using the same argument as in the proof of Theorem 1.2 of Rao and Rao[19], we obtain that z is a unique common fixed point of A and B (in this part of the proof it suffices to employ only condition (22)).

Remark 1 : By putting in Theorem 4.3 $A = B$ and replacing $m(x, y)$ by $d(x, y)$ in (21), we get Theorem B of Rao and Rao[18].

Remark 2 : Theorem 4.3 can be extended to three maps in a way similar to that of Rao and Rao[19] (Theorem 1.2).

Remark 3 : It is also possible to get an extension of Theorem 1.3 of the above authors[19]. One should replace $m(x, y)$ by $M(x, y)$ in our Theorem 4.3 and assume the continuity of the involved maps. Such a way of treatment was also indicated by Jungck *et al.*[10].

5. A Fixed Point Theorem for Sequences of Maps

We shall present here how our Theorem 3.3 can be extended to a sequence of maps. Most of the authors use an iteration procedure involving all of the considered maps (see e.g. Chaterji[5], Rhoades *et al.*[22] and Sessa *et al.*[23]). Following Jungck *et al.*[10] and Rhoades[21] we shall use an iteration involving the maps S, T, A_i and A_j with suitably fixed integers i and j. It will enable us to use different contractive gauge functions similar to that in Corollary 1 of Rhoades[21].

Theorem 5.1 — Let S and T be selfmaps of a complete metric space (X, d) and either S or T is continuous. Let $\{A_i\}_{i=0}^{\infty}$ be a sequence of selfmaps of X satisfying

(i) $A_0(X) \subseteq T(X)$ and $A_i(X) \subseteq S(X)$ for $i \in \mathbb{N}$;

(ii) pairs of (A_0, S) and (A_i, T) $(i \in \mathbb{N})$ are compatible;

(iii) for each $i \in \mathbb{N}$ there exists an upper semicontinuous function $\Phi_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\Phi_i(t) < t$ for all $t > 0$ and, for any $x, y \in X$, $d(A_0x, A_iy) \leq \Phi_i\,[M_i(x, y)]$ where $M_i(x, y) := \max\,\{d(Sx, Ty),\ d(A_0x, Sx),\ d(A_iy, Ty),\ [d(A_0x, Ty) + d(A_iy, Sx)]/2\}$.

Then all the $A_i\ (i \in \mathbb{N} \cup \{0\})$, S and T have a unique common fixed point.

Proof : From Theorem 3.3, for each $i \in \mathbb{N}$, A_0, A_i, S and T have a unique common fixed point z_i. Suppose $z_i \neq z_{i+1}$ for some $i \in \mathbb{N}$. Putting in the inequality of (iii) $x = z_{i+1}$ and $y = z_i$ we obtain that $d(A_0 z_{i+1}, A_i z_i) \leq \Phi_i\,[M_i\,(z_{i+1}, z_i)]$. Since $A_0 z_{i+1} = z_{i+1}$, $A_i z_i = z_i$, $M_i\,(z_{i+1}, z_i) = d(z_i, z_{i+1})$ and $\Phi_i(t) < t$ for all $t > 0$, we get that $d(z_i, z_{i+1}) \leq \Phi_i\,[d(z_i, z_{i+1})] < d(z_i, z_{i+1})$, a contradiction. Thus $z_i = z_{i+1}$ for all $i \in \mathbb{N}$ which means that the sequence $\{z_i\}$ is constant. Hence z_1 is the unique common fixed point of $A_i\ (i \in \mathbb{N} \cup \{0\})$, S and T.

Remark 1 : By putting in Theorem 5.1 $S = T = J$ and $\Phi_i\,(t) = ht$ for some h in $[0, 1)$, $i \in \mathbb{N}$ and $t \in \mathbb{R}_+$, we obtain the theorem of Chaterji[5].

Remark 2 : By putting in Theorem 5.1 $\Phi_i = \Phi$ for $i \in \mathbb{N}$, where $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing and upper semicontinuous function such that $\Phi(t) < t$ for all $t > 0$, we obtain a theorem which is still more general than the sufficiency part of Theorem 3 by Sessa *et al.*[23] because we use compatible maps instead of weakly commuting and we require the contractive condition to hold only for pairs $(0, j)$, $(j \in \mathbb{N})$, while Sessa *et al.*[23] require it to hold for all pairs (i, j) with $i, j \in \mathbb{N}$ and $i \neq j$.

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