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SEQUENCES IN NON-ARCHIMEDEAN LOCALLY CONVEX SPACES

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The main goal of this article is to introduce and study the non-Archimedean versions of the c-sequential and s-bornological spaces. The new classes are located in the context of the locally $I\!K$-convex classes already known.

1. INTRODUCTION

In this paper we study the $I\!K$-c-sequential and $I\!K$-s-bornological spaces, the non-Archimedean analogues of the classical c-sequential and s-bornological spaces, which are studied by Snipes[7] and Webb[10]. We discuss its characterizations and permanence properties. We also consider a sufficient condition for a locally $I\!K$-convex space to be $I\!K$-s-bornological through the Mackey-τ-closed graph theorem. For this purpose, we shall adopt the notation and terminology of Van Tiel[8] and Caldas[1].

Throughout this paper, $I\!K$ stands for a valued field with a non trivial valorization $|\cdot|$. All locally $I\!K$-convex spaces will be assumed Hausdorff. If E and F are locally $I\!K$-convex spaces, $\mathcal{L}(E, F)$ (resp. $\mathcal{L}_b(E, F)$), $\mathcal{L}_{sc}(E, F)$) will denote the vector space of all continuous (resp. bounded over the bounded sets, sequentially continuous) linear maps from E into F. When $F = I\!K$ we shall adopt the simpler notation E' to denote $\mathcal{L}(E, I\!K)$ (resp. E_b' to denote $\mathcal{L}_b(E, I\!K)$, E_{sc}' to denote $\mathcal{L}_{sc}(E, I\!K)$).

Lemma 1.1 — Let E and F be two locally $I\!K$-convex spaces. Then every sequentially continuous linear mapping from E into F is locally bounded.

PROOF : Let $f \in \mathcal{L}_{sc}(E, F)$ and $\mu \in I\!K$ fixed with $|\mu| > 1$. Then the scalar sequence $\lambda_n = \mu^n$ ($n \in I\!N^*$) is such that $|\lambda_0| < |\lambda_1| . . . < |\lambda_n| < . . .$ and $\lim_{n \to \infty} |\lambda_n| = \infty$. Suppose that $f \notin \mathcal{L}_b(E, F)$. Then there exists a bounded subset B of E such that $f(B)$ is not bounded in F. Therefore, for a balanced neighbourhood V of zero in F, for each $n \in I\!N^*$, we can find $x_n \in B$ with $f(x_n) \notin \lambda_n^2 V$. The sequence $y_n = x_n/\lambda_n$ converges to zero in E although $(f(y_n))_{n \in I\!N^*}$ is not bounded in F, since the point $f(y_n)$ are not in $\lambda_n V$. This contradicts our first assumption. Hence

$f \in \mathcal{L}_b(E, F)$. □

Corollary 1.1 — Let E and F be two locally $\mathbb{K}$-convex spaces. Then

(i) $\mathcal{L}(E, F) \subset \mathcal{L}_{sc}(E, F) \subset \mathcal{L}_b(E, F)$;

(ii) If E is $\mathbb{K}$-bornological, we have $\mathcal{L}(E, F) = \mathcal{L}_{sc}(E, F) = \mathcal{L}_b(E, F)$.

In this way, for locally $\mathbb{K}$-convex spaces and clearly for n.a. normed spaces we have the following properties :

If E is a $\mathbb{K}$-bornological space, F a locally $\mathbb{K}$-convex space and f a linear mapping from E into F then

(i) f is continuous if sequentially continuous;

(ii) f is sequentially continuous if locally bounded.

For this, we introduce the following notions :

Definition 1.1 — (i) A locally $\mathbb{K}$-convex space E is said to be convex-sequential (or $\mathbb{K}$-c-sequential) if, for each locally $\mathbb{K}$-convex space F, we have

$$\mathcal{L}(E, F) = \mathcal{L}_{sc}(E, F);$$

(ii) A locally $\mathbb{K}$-convex space E is said to be sequentially bornological (or $\mathbb{K}$-s-bornological) if, for each locally $\mathbb{K}$-convex space F, we have

$$\mathcal{L}_{sc}(E, F) = \mathcal{L}_b(E, F).$$

The following theorem is an easy consequence of the Definition 1.1 and Corollary 1.1.

Theorem 1.1 — A locally $\mathbb{K}$-convex space E is $\mathbb{K}$-bornological if and only if E is both $\mathbb{K}$-c-sequential and $\mathbb{K}$-s-bornological.

2. Some Characterizations

Definition 2.1 — A set V in a locally $\mathbb{K}$-convex space is called a sequential neighbourhood of zero if every sequence converging to zero belongs to V eventually, i.e., if $x_n \to 0$ implies $x_n \in V$ for all sufficiently large n.

Theorem 2.1 — Let E be a locally $\mathbb{K}$-convex-space. Then the following statements are equivalent :

(i) E is $\mathbb{K}$-c-sequential;

(ii) Every $\mathbb{K}$-convex sequential neighbourhood of zero in E is a neighbourhood of zero.

Proof : (i) ⇒ (ii) Let V be a $\mathbb{K}$-convex sequential neighbourhood of zero in E. Then V is absorvent. For if $x \in E$, then $x/\lambda_n \to 0$ in E, with $(\lambda_n)_{n \in \mathbb{N}}$ as in Lemma 1.1. So there exists $n_0 \in \mathbb{N}^*$ such that $x/\lambda_n \in V$ for all $n \geq n_0$. Choosing $\alpha_{n_0} = 1/\lambda_{n_0}$ in $\mathbb{K}^*$, we have $\alpha x \in V$ for all $\alpha \in \mathbb{K}$ with $|\alpha| < |\alpha_{n_0}|$. Now, let $p_V : E \to \mathbb{R}_+$ be the n.a. seminorm defined by $p_V(x) = \inf\ \{|\lambda| : \lambda \in \mathbb{K}^*,\ x \in \lambda V\}$

for all $x \in E$ (Van Tiel[8]). We denote by E_V the vector space E endowed with the locally $I\!K$-convex topology defined by p_V. We shall prove that V is a neighbourhood of zero in E. For this purpose it suffices to prove that the mapping $I : E \to E_V$ is continuous. Suppose that $x_n \to 0$ in E. Then for each $\lambda \in I\!K^*$, with $|\lambda| > 1$, there is $n_0 \in I\!N^*$ such that $x_n \in \lambda V$, for all $n \geq n_0$. Hence and by definition of $E_V, x_n \to 0$ in E_V. Since E is $I\!K$-c-sequential, I is continuous and (ii) is proved.

(ii) $\Rightarrow$ (i) Let F be a locally $I\!K$-convex space and $f : E \to F$ a sequentially continuous linear mapping. If V is a neighbourhood of zero in F, then $f^{-1}(V)$ is a sequential neighbourhood of zero in E, hence a neighbourhood of zero. Therefore f is continuous and E is a $I\!K$-c-sequential space. □

Theorem 2.2 — Let E be a locally $I\!K$-convex space. Then the following conditions are equivalent :

(i) E is $I\!K$-s-bornological;

(ii) Every $I\!K$-convex bornivorous subset of E is a sequentially neighbourhood of zero;

(iii) Every n.a. seminorm p on E which is locally bounded (i.e., such that $\{x \in E : p(x) < 1\}$ is a bornivorous set) is sequentially continuous.

PROOF : (i) $\Rightarrow$ (ii) Let B be a $I\!K$-convex bornivorous subset of E. Hence B is absorbent and $p_B : x \in E \to \inf\{|\lambda| : \lambda \in I\!K^*, x \in \lambda B\}$ is a n.a. seminorm on E. To guarantee that B is a sequential neighbourhood of zero in E, it is enough to prove that $I : E \to (E, p_B)$ is sequentially continuous, since for $x_n \to 0$ in E there will be $n_0 \in I\!N^*$ such that $p_B(x_n) < 1$ for all $n \geq n_0$ and then, $x_n \in B$ for all $n \geq n_0$. To prove our assertion, it is enough to show that I is locally bounded, since E is $I\!K$-s-bornological. Let A be a bounded subset of E. There exists $\lambda \in I\!K^*$ such that $A \subset \lambda B$ and consequently $p_B(x) \leq |\lambda|$ for all $x \in A$. Thus A is bounded in (E, p_B).

(ii) $\Rightarrow$ (iii) Let p be a n.a. locally bounded seminorm on E. Then $V_p = \{x \in E : p(x) < 1\}$ is a $I\!K$-convex bornivorous set and, therefore, a sequential neighbourhood of zero in E, by hypothesis. We must show that p is sequentially continuous. Let $(x_n)_{n \in I\!N^*}$ be a sequence in E which converges to zero and let $\varepsilon > 0$ be given. Let $\mu \in I\!K$ with $0 < |\mu| < \varepsilon$. Then $x_n/\mu \to 0$ in E. So there is $n_0 \in I\!N^*$ such that $x_n/\mu \in V_p$ for all $n \geq n_0$. Then $p(x_n/\mu) < 1$, i.e., $p(x_n) < |\mu| < \varepsilon$, for all $n \geq n_0$. Thus p is sequentially continuous.

(iii) $\Rightarrow$ (i) Let F be a locally $I\!K$-convex space and let $f : E \to F$ be a locally bounded linear mapping. If p is any element of the family of (continuous) seminorms defining the topology of $F, p \circ f$ is a n.a. seminorm on E which is locally bounded and therefore, sequentially continuous, by hypothesis. Thus f is sequentially continuous. Consequently E is $I\!K$-s-bornological. □

Theorem 2.3 — Let E be a locally $I\!K$-convex space. Then the following statements are equivalent :

(i) E is $I\!K$-c-sequential;

(ii) If a locally $I\!K$-convex space F and a set M of linear mappings of E into F are given such that, for each sequence $(x_n)_{n \in I\!N^*}$ in E converging to zero and neighbourhood V of zero in F, there is $n_0 \in I\!N^*$ such that $f(x_n) \in V$ for all $n \geq n_0$, $f \in V$, then M is equicontinuous.

PROOF : (i) $\Rightarrow$ (ii) Let E be $I\!K$-c-sequential and assume that M has the property indicated in part (ii). We claim that for each $I\!K$-convex neighbourhood V of zero in F, $\bigcap_{f \in M} \{f^{-1}(V)\}$ is a neighbourhood of zero in E. By Theorem 2.1, it is enough to show that $\bigcap_{f \in M} \{f^{-1}(V)\}$ is a $I\!K$-convex sequential neighbourhood of zero. Indeed, if $x_n \to 0$ in E there is $n_0 \in I\!N^*$ such that $f(x_n) \in V$ for all $n \geq n_0, f \in M$, that is, such that $x_n \in \bigcap_{f \in M} \{f^{-1}(V)\}$ for all $n \geq n_0$. Our assertion is proved.

(ii) $\Rightarrow$ (i) Assume (ii). Let F be a locally $I\!K$-convex space and let $f : E \to F$ be a sequentially continuous linear mapping. Then $M = \{f\}$ satisfies the condition required in (ii), so it is equicontinuous. Thus f is continuous. This proves that E is $I\!K$-c-sequential. □

Theorem 2.4 — Let E be a locally $I\!K$-convex space. Then :

(i) E is $I\!K$-c-sequential if and only if, for each n.a. normed space F, each sequentially continuous linear mapping from E into F is continuous.

(ii) E is $I\!K$-s-bornological if and only if, for each n.a. normed space F, each locally bounded linear mapping from E into F is sequentially continuous.

PROOF : We shall prove the case (i), the other being analogous. Necessity follows immediately. Let us turn to sufficiency. Let V be a $I\!K$-convex sequential neighbourhood of zero in E. Then V is absorbent. As in the proof of Theorem 2.2, we obtain a n.a. seminorm p_V which is sequentially continuous. So $N = \{x \in E : p_V(x) = 0\}$ is a linear subspace of E. Let us consider the n.a. normed space $(E/N, |\cdot|_V)$, where $|\cdot|_V : E/N \to I\!R_+$ is defined by the relation $|\dot{x}|_V = p_V(x)$ for all $\dot{x} \in E/N$. Let $\pi : x \in E \to \dot{x} \in E/N$ be the canonical linear mapping from E onto E/N. We claim that π is sequentially continuous. For if $(x_n)_{n \in I\!N^*}$ converges to zero in E, $(p_V(x_n))_{n \in I\!N^*}$ converges to zero and so does $(|\pi(x_n)|_V)_{n \in I\!N^*} = (|\dot{x}_n|_V)_{n \in I\!N^*} = (p_V(x_n))_{n \in I\!N^*}$. Consequently $(\pi(x_n))_{n \in I\!N^*}$ converges to $N = \pi(0)$ and our assertion is proved. By hypothesis, π is continuous. It follows that the inverse image of the neighbourhood $B = \{\dot{x} \in E/N : p_V(x) < 1\}$ of zero in $(E/N, |\cdot|_V)$ under π, namely $\pi^{-1}(B) = \{x \in E : p_V(x) < 1\}$, is a neighbourhood of zero in E. Since $\pi^{-1}(B) \subset V$ (Van Tiel[8]), the set V is a neighbourhood of zero in E. By Theorem 2.1, E is a $I\!K$-c-sequential space. □

3. PERMANENCE PROPERTIES

Proposition 3.1 — Let E be a vector space, $(E_i)_{i \in I}$ a family of $I\!K$-c-sequential spaces and for each index $i \in I$, let f_i be a linear map from E_i into E. Then E

endowed with the final locally $\mathbb{K}$-convex topology for the family $(E_i, f_i)_{i \in I}$ is a $\mathbb{K}$-c-sequential space.

PROOF : Let F be a locally $\mathbb{K}$-convex space and $f \in \mathcal{L}_{sc}(E, F)$. Since each f_i is continuous, $f \circ f_i$ is sequentially continuous and so continuous by hypothesis, for every $i \in I$. Therefore $f \in \mathcal{L}(E, F)$.

Corollary 3.1 — The class of all $\mathbb{K}$-c-sequential spaces is stable under the formation of separated quotients, arbitrary direct sums and inductive limits.

Lemma 3.1 — Let $(E_i)_{i \in I}$ be a family of locally $\mathbb{K}$-convex spaces and let $(F, |\cdot|_F)$ be a n.a. normed space. Suppose that $f: E = \Pi_{i \in I} E_i \to F$ is a locally bounded linear mapping when E is endowed with the product topology. Then there exists a finite subset A of I such that $f_{|\bar{E}_i} = 0$ for all $i \notin A$, where $\bar{E}_i = \{(0, ..., 0, x_i, 0, ...) \in E : x_i \in E_i\}$.

PROOF : If not, there exists a countable infinite subset of I, $\{i_1, i_2, ..., i_n, ...\}$, such that each $f_{|\bar{E}_{i_j}}$ is not zero. For every $j \in \mathbb{N}^*$, let $(0, ..., x_i, 0, ...) \in \bar{E}_{i_j}$ be such that $f(0, ..., 0, x_{i_j}, 0, ...) \neq 0$. By taking $(\lambda_j)_{j \in \mathbb{N}^*}$ a sequence in $\mathbb{K}^*$ with $\lim_{j \to \infty} |\lambda_j| = \infty$ and

$$a_j = \frac{(0, ..., 0, \lambda_j x_{i_j}, 0, ...)}{|f(0, ..., 0, x_{i_j}, 0, ...)|_F} \in \bar{E}_{i_j} \qquad (j \in \mathbb{N}^*)$$

it follows that $(a_j)_{j \in \mathbb{N}^*}$ is a bounded sequence with

$$\left| f\left(\frac{(0, ..., 0, \lambda_j x_{i_j}, 0, ...)}{|f(0, ..., 0, x_{i_j}, 0, ...)|_F}\right)\right|_F = |\lambda_j| \cdot$$

This implies that f is not locally bounded, contradicting our assumption. □

Proposition 3.2 — Let $(E_n)_{n \in \mathbb{N}^*}$ be a countable family of locally $\mathbb{K}$-convex spaces and $E = \Pi_{n \in \mathbb{N}^*} E_n$ be equipped with the product topology. Then

(i) E is $\mathbb{K}$-c-sequential if and only if each E_n is $\mathbb{K}$-c-sequential.

(ii) E is $\mathbb{K}$-s-bornological if and only if each E_n is $\mathbb{K}$-s-bornological.

PROOF : Let F be a n.a. normed space. For a fixed $n_0 \in \mathbb{N}^*$, let $f : E_{n_0} \to F$ be a sequentially continuous linear mapping. We define $\bar{f}: E \to F$ by $\bar{f}(x) = f(x_{n_0})$, if $x = (x_n)_{n \in \mathbb{N}^*}$, i.e., $\bar{f} = f \circ P_{n_0}$, where P_{n_0} denotes the projection of E onto E_{n_0}. It is clear that $\bar{f}$ is a sequentially continuous linear mapping. Therefore $\bar{f}$ is continuous by hypothesis and consequently, f is continuous. By Theorem 2.4, E_{n_0} is $\mathbb{K}$-c-sequential. Conversely, let $f: E \to F$ be a sequentially continuous linear mapping, where F is an arbitrary n.a. normed space. By Lemma 1.1, f is locally bounded and then, by Lemma 3.1, there exists a finite subset A of $\mathbb{N}^*$ such that $f_{|\bar{E}_n} = 0$ for all $n \in \mathbb{N}^*, n \notin A$ ($\bar{E}_n$ is defined as in Lemma 3.1). Therefore, it is enough to prove the assertion for a finite product of $I\!K$-c-sequential spaces, which is clearly a consequence of Corollary 3.1. This proves (i). Analogously, one proves (ii). □

Proposition 3.3 — Suppose that $I\!K$ is spherically complete. Let E be a locally $I\!K$-convex space and L be a proper sequentially dense subspace of E. If L is $I\!K$-c-sequential then E is $I\!K$-c-sequential.

PROOF : Let f be a sequentially continuous linear form on E. Then the restriction f_L of f to L is continuous by hypothesis. The n.a. version of the Hahn-Banach theorem[4] assures the existence of a continuous linear mapping g from E into $I\!K$ such that $g_L = f_L$. Hence $g = f$ on E since L is sequentially dense in E, which concludes the proof. □

4. A SUFFICIENT CONDITION FOR $I\!K$-s-BORNOLOGICITY THROUGH THE CLOSED GRAPH THEOREM

Definition 4.1[5] — A sequence $(x_n)_{n\in I\!N^*}$ of elements of a locally $I\!K$-convex space E is said to converge to $x_0 \in E$ in the Mackey sense (or to be locally convergent to x_0) if there is a bounded $I\!K$-convex subset B of E so that $(x_n)_{n\in I\!N^*}$ converges to x_0 in the normed space (E_B, p_B). A convergent sequence in the Mackey sense is always convergent.

Definition 4.2 — Let (E, τ_E) and (F, τ_F) be two locally $I\!K$-convex spaces and f be a mapping from E into F. We say that f has an M–τ_F-closed graph if, for each sequence $(x_n)_{n\in I\!N^*}$ converging to x_0 in the Mackey sense in (E, τ_E), we have that $(f(x_n))_{n\in I\!N^*}$ converges to $f(x_0)$ in (F, τ_F).

Theorem 4.1 — Let (E, τ_E) be a locally $I\!K$-convex space. Suppose that for every n.a. normed space $(F, |\cdot|_F)$, every linear mapping $f: E \to F$ whose graph is M-$|\cdot|_F$-closed is sequentially continuous. Then (E, τ_E) is $I\!K$-s-bornological.

PROOF : Let V be a $I\!K$-convex bornivorous subset of E and let π and $(E/N, |\cdot|_V)$ be as in Theorem 2.4. We claim that π has an M-$|\cdot|_V$-closed graph. For then, by hypothesis, π will be sequentially continuous and, reasoning as in the proof of Theorem 2.4, we will have that E is $I\!K$-s-bornological. To prove the above assertion, let $(x_n)_{n\in I\!N^*}$ be a sequence in (E, τ_E) that converges to zero in the Mackey sense. By Definition 4.1, there is a τ_E-bounded $I\!K$-convex subset B such that $(x_n)_{n\in I\!N^*}$ converges to zero in the n.a. normed space E_B. Since V is τ_E-bornivorous, there exists $\lambda \in I\!K^*$ such that $B \subset \lambda V$. Then, for $\varepsilon > 0$ given, there is $n_0 \in I\!N^*$ such that $p_B(x_n) < \varepsilon/|\lambda|$ for all $n \geq n_0$. Therefore $|\pi(x_n)|_V = p_V(x_n) \leq |\lambda| p_B(x_n) < \varepsilon$ for all $n \geq n_0$. Hence $(\pi(x_n))_{n\in I\!N^*}$ converges to zero in $(E/N, |\cdot|_V)$. This proves that π has an M-$|\cdot|_V$-closed graph. □

We must observe that the Theorem 4.1 is known when (E, τ_E) is a d-$I\!K$-barrelled space[2].

Finally, we shall see in the following example that it is not true that every $I\!K$-c-sequential space in $I\!K$-s-bornological. Before proving this, we need some preliminary results, analogous to that obtained by Hampson and Wilansky[3].

Let $\mathcal{U}_{cs}$ be the class of all $I\!K$-convex sequential neighbourhoods of zero in (E, τ_E).

Proposition 4.1 — (i) There is a locally $I\!K$-convex Hausdorff topology τ_{cs} on E for which $\mathcal{U}_{cs}$ is a base of neighbourhoods of zero.

(ii) τ_{cs} is the finest locally $I\!K$-convex Hausdorff topology on E which has the same convergent sequences as τ_E.

(iii) (E, τ_{cs}) is a $I\!K$-c-sequential space.

PROOF : The proof of (i) is clear.

(ii) Clearly $\tau_E \leq \tau_{cs}$, so τ_{cs} is a Hausdorff topology and every sequence τ_{cs}-convergent in E is τ_E-convergent. On the other hand, let U be a τ_{cs}-neighbourhood of zero in E. Then $U \supset V$, for some $V \in \mathcal{U}_{cs}$ and consequently, given $x_n \to 0$ in (E, τ_E), there exists $n_0 \in I\!N^*$ such that $x_n \in U$ for all $n \geq n_0$. Therefore $x_n \to 0$ in (E, τ_{cs}). Thus τ_E and τ_{cs} have the same convergent sequences. Now suppose that σ is another locally $I\!K$-convex Hausdorff topology on E such that (E, σ) has the same convergent sequences as (E, τ_E). If V is a σ-neighbourhood of zero in E, then $V \in \mathcal{U}_{cs}$. Thus V is a τ_{cs}-neighbourhood of zero in E. Hence $\sigma \leq \tau_{cs}$. So τ_{cs} is the finest such topology.

(iii) It is immediate from (ii) and Theorem 2.1. □

Corollary 4.1 — $(E, \tau_{cs})' = (E, \tau_{cs})'_{cs} = (E, \tau_E)'_{cs}$.

From now on, we assume that $I\!K$ is spherically complete and use $\tau_c(E, E')$ to denote the strongest (E, E')-compatible topology on the locally $I\!K$-convex space E (see Van Tiel[8]). We also write η instead of $\sigma(E', E)$.

Corollary 4.2 — A locally $I\!K$-convex space (E, τ_E) is $I\!K$-c-sequential if the topology τ_E coincides with $\tau_c(E, E')$ and each sequentially continuous linear mapping from E into $I\!K$ is continuous.

PROOF : It follows from Proposition 4.1. that $\tau_E \leq \tau_{cs}$. By hypothesis and Corollary 4.1, $(E, \tau_{cs})' = (E, \tau_E)'$. Therefore the topology τ_{cs} is (E, E')-compatible and so $\tau_E \geq \tau_{cs}$ (Van Tiel[8]). Then $\tau_E = \tau_{cs}$ and E is $I\!K$-c-sequential. □

Proposition 4.2 — Let (E, τ_E) be a locally $I\!K$-convex space such that each η-sequentially continuous linear mapping from E' into $I\!K$ is η-continuous. Then η_{cs} is (E, E')-compatible.

PROOF : We claim that $(E', \eta_{cs})' = E$. Indeed, let $f \in (E', \eta_{cs})'$. Then f is η_{cs}-sequentially continuous and so η-sequentially continuous. By our hypothesis, f is η-continuous. Thus $f \in (E', \eta)' = E$. On the other hand, $E \subset (E', \eta_{cs})$, since $\eta \leq \eta_{cs}$. □

Proposition 4.3 — Let (E, τ_E) be a separable and complete locally $I\!K$-convex space. Then every η-sequentially continuous linear mapping from E' into $I\!K$ is η-continuous.

PROOF : Let $f: E' \to I\!K$ be a η-sequentially continuous linear mapping and $\mathcal{U}$ a fundamental system of neighbourhoods of zero in E. For each $U \in \mathcal{U}$, the pseudo-polar $U^p = \{g \in E' : \sup_{x \in U} |g(x)| < 1\}$ of U in E' (Van Tiel)[9] is then equicontinuous and therefore $(U^p, \eta|_{U^p})$ is metrizable. Hence $f|_{U^p}$ is $\eta|_{U^p}$-continuous for each $U \in \mathcal{U}$. Since E is complete, f is η-continuous in E'. □

Proposition 4.4 — Let (E, τ_E) be a non-reflexive n.a. Banach space. Then $(E', \tau_c(E', E))$ is not $I\!K$-bornological.

PROOF : For $(E', \tau_c(E', E))$ is a semireflexive space which is not reflexive, since its strong dual (E, τ_E) is not reflexive (Van Tiel[8]). Therefore $(E', \tau_c(E', E))$ is not $I\!K$-infrabarrelled, hence not $I\!K$-bornological. □

Example 4.1 — Let E be a separable non-reflexive n.a. Banach space. By Proposition 4.1(iii), (E', η_{cs}), is $I\!K$-c-sequential. Now suppose that (E', η_{cs}) has the following property :

$$(E', \eta_{cs})'_b = (E', \eta_{cs})'_{sc}. \qquad \dots (1)$$

By Proposition 4.3, $(E', \eta)'_{sc} = (E', \eta)'$. Hence and by Proposition 4.2, η_{cs} is (E', E)-compatible. If we put $\tau = \tau_c(E', E)$ then

$$(E', \tau)' \subset (E', \tau)'_b = (E', \eta_{cs})'_b = (E', \eta_{cs})'_{sc} = (E', \eta_{cs})' \subset (E', \tau)',$$

i.e., $(E', \tau)'_b = (E', \tau)'$. Therefore, by (Van Tiel[8]), (E', τ) is $I\!K$-bornological, contradicting Proposition 4.4. Then (E', η_{cs}) has not the property (1). Thus (E', η_{cs}) is not $I\!K$-s-bornological.

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